Alexey Milyutin was born in Moscow on the 27th of July 1925. His father, also Alexey Milyutin, was a Communist Party official in the 20s, later he worked as an editor at a radio station. His uncle, Vladimir Milyutin, was Minister of Agriculture with the new government of the Russian Republic. His mother, Olga Veiland, like her husband, was an active figure of the Communist Party. During the time of the Civil War and peace she was involved in the Party and government work, later she worked for the *Rabotnitsa* magazine for many years.

Before 1937, Alexey Milyutin devoted most of his time to music and seriously considered pursuing a musical career. But in 1937 he had to quit music as the tragic events of that year hit the Milyutin family among many others. Music remained Milyutin’s love throughout his life.

After the Great Patriotic War burst out, the Milyutin family was evacuated from Moscow to the Ryazan Region and then to Tatarstan. It was in the town of Oktash, Tatarstan, that Milyutin graduated from the 9th grade of high school. The Milyutins came back to Moscow in 1942. Alexey was enrolled for a course at the Moscow State University, where he completed his high school education. The course administrators initially promised to admit highly performing course graduates at the University without exams (at that time, there was no issue of competition). Alexey Milyutin completed the course with good results; however the promise was not kept and he was not automatically enrolled. With a great deal of assertiveness, Alexey approached I.G. Petrovsky with an application for admission at the Department of Mechanics and Mathematics (DMM). After several meetings and discussions with Ivan Petrovsky, Milyutin was enrolled as a DMM student without exams.

In 1948, Alexey Milyutin successfully completed his studies at the DMM and was recommended as a post-graduate student to Prof. V.V. Nemytsky. Milyutin’s independence as a scientist showed early. The area of research he chose for his thesis was such that his research superviser could be of little help. The topic was prompted by the issue which was discussed in the DMM hallways and which Alexey Milyutin beleived was only really interesting to himself. He didn’t have the slightest idea that he was actually working on S.Banach’s very difficult problem. B.Mityagin tells about it in his introduction to A. Pelczinsky’s book, *Linear Continuations, Linear Averaging, and Their Applications,* ‘C(S)-spaces of continuous functions at S-compacts is a classical subject which has been studied from different points of view in topology, functional analysis, measure theory, harmonic analysis. A simple excample of close ties between algebraic properties of C(S) and topological properties of the S-compact is the following statement, “algebras C(S1) and C(S2) are isomorphic when and only when the compacts S1 and S2 are homeomorphic”. It is not difficult to think of many other such examples, which will be deeper and more informative. That made Milyutin’s solution of 1951 even more unexpected; he established linear isomorphism between the spaces of continuous functions in a segment and a square. Besides, ironically (was it by chance or on purpose?) for 15 years after that, the result remained unpublished and actually unknown to many mathematicians who had been trying to solve the same problem. But there’s no great loss without some small gain: even after it was solved, the problem continued to provoke more research, which among other things led to the solution of Banach’s other problem, proof of topological equivalnce (homeomorphism) of all Banach’s separable spaces (M.I. Kadets, C. Bessag, A. Pelczinsky, V. Klee).

Publication of results was not a neccessary condition of formal approval of the thesis at the time. The thesis was defended in 1951, with M.I Gelfand and L.A. Lyusternik as reviewers. The work remained unpublished. The issue of the equivalence of spaces of continuous functions was raised again at the International Mathematical Congress in Moscow in 1966. Luckily, the manuscript survived and the result was presented and later published, largely due to the initiative and efforts by mathematicians from Kharkov and Poland. After such a successful start, Alexey Milyutin moved towards a different area of research, which covered some most important applied problems of the 50s and 60s. Y.M. Kazhdan writes about that time, ‘In the summer of 1952, A.A. Milyutin was brought into a computation group at the Institute of Physical Problems. The group was created by L.D Landau, a full member of the Academy of Sciences, to run nuclear-related computations. A.A. Milyutin had no previous experience with numerical methods of problem solving. His mathematical culture, however, helped him get the feel of the existing, albeit limited, experience in the field rather quickly. Of special importance at the time were difference methods of solving problems in partial derivatives, especially issues related to the stability of the methods. A.A. Milyutin took an active part in the learning and further improvement of neccessary tecniques. One has to mention a dramatic (as it seemed then) episode connected with Milyutin’s work. As it happens, the problems to be computed came from a group of physicists led by L.D. Landau. Well, after long computations (which then were made by electromechanical Mercedeses) and sophisticated analysis Milyutin discivered a mathematical mistake in the problem statement, which absolutely shocked the physicists. It took Landau’s genius to assess the negative impact of the mistake. Luckily the mistake was found to have no impact on the final result. In the ensuing years, as a research fellow at the Institute of Chemical Physics (ICP), Academy of Sciences of the USSR, A.A. Milyutin continued to develop methods and explore problems related to the equations of mathematical and chemical physics.’

Here is what A.M. Kogan, a former member of ICP, said about a later period, ‘In 1954, a computation group was created at ICP on the initiative of N.N. Semenov, the Institute’s Director. The group creation process (interviewing, professional screening, HR approvals, etc) was a responsibility of L.A. Chudov, the first leader of the group, and A.A. Milyutin, who had brought in impressive computational experience from the Institute of Physical Problems. The new group was made up of a few young Moscow State University graduates, who had been trained in pure mathematics and had no idea of methods, let alone “tricks” of computation. The burden of training the group members in computation was almost fully borne by A.A. Milyutin. That task was further complicated by the lack of time. At the very beginning, the group was asked to do computations for problems the ICP leaders were interested in, such as moderation of fast neutrons in substances modelling the structure of human body tissues, propagation of strong explosions in various media, computation of real gas state equations, for example the state of air at high temperatures, polymer chain statistics, etc. It was from Milyutin that the group members learnt computational basics, and it was under his leadership that they explored the above-mentioned problems and mastered their computational skills from a practical assignment. A.A. Milyutin generously shared his unique computational experience (which could not be found in the existing literature) with his colleagues. For example, he always emphasized the importance of physical and chemical (and not just mathematical) considerations for a high-quality computation.

Even at the problem statement stage, where besides equations one has to formulate additional conditions, for example initial or boundary conditions or asymptotic at infinity, he was not just guided by the assessment of the validity of the probable theorem on the unique existence of equations with additional conditions, proposed by the physicists based on their knowledge of the process properties, including verified properties, but tried to understand their nitty-gritty. By analyzing the situation and paraphrasing the statement, he would finally come to the correct statement evolving the solution sought by the physicists, as the computation process should allow for physical laws and properties of the phenomenon under exploration; in some instances that leads to the unambiguously most reliable algorithm.

Another example is A.A. Milyutin's approach to the development of the solution method which he referred to as the 'trivialization' of original equations. What was used was not just their simplified model – it was maximum possible simplification which retained the general properties of the original equations, if possible the most material properties. An algorithm was developed for the simplified problem and generalized to the original one. A.A. Milyutin had a special gift for the trivialization which 'did not throw out the child'. His approach was very impressive and often highly effective.

After L.A. Chudov left ICP A.A. Milyutin was appointed as the group leader. Milyutin's group streamlined computational support for ICP laboratories. It was a time when the assignments grew in number and, dramatically, in complexity. It was also a period of transition to computer solution of the whole array of problems. The problems covered a great variety of subjects. They included numerous examples of chemical reaction kinetics, burning and explosion processes, computations related to chemical reactors, gas dynamics with strong shock waves, strong explosions in nonhomogenious atmosphere, electromagnetic radiation transport in the air, computation of volt-ampere characteristics in electrical circuits containing chemical solutions, etc.

It so happened then that Milyutin's group went beyond the Institute's limits up to a national, even international, level. At the time, the USSR and USA sent their representatives to a conference in Geneva to discuss a ban for nuclear weapon tests in all environments. The conference was halting because unlike Soviet experts, the Americans were convinced that nuclear explosion signals could be confidently distinguished from natural phenomena only at a small distance from the venue of a possible breach of the future treaty. So the Americans were asking for permission to deploy radars in the USSR, which the Soviet side couldn't agree to.

Milyutin's group was given an assignment to calculate nuclear explosion effects under two scenarios, in the air and in a spherical cavity in the ground. The "air"-related calculations were tied to a special rf pulse discovered by ICP scientists. The pulse is radiated at the time of a nuclear explosion. The calculation showed that even at long distances, a nuclear explosion can be confidently distinguished from other bursts or explosions.

In the "ground" case, the calculations had to cover a period between the formation of a strong shock wave in the cavity center as a result of a nuclear explosion and the sound stage of the flow; that included numerous wave reflections from the cavity walls and their collapse in the center, a very difficult task for the time. The air impact on the cavity walls throughout this long period of time was included as an additional condition for a problem about geophysical wave propagation along the earth's surface. As a result, it became possible to confidently distinguish explosion characteristics from similar characteristics of earthquakes, even at long distances. When these results were presented to the American experts it turned out that they were not familiar with such reasoning. Some time later, the Americans ran analogous calculations and received the same results. The Treaty was signed without any foreign radars in the Soviet Union.

After considerable expansion of the computational division at ICP, acquisition of a powerful computer and the appointment of A.Y. Povzner as head of the Mathematical Department, A.A. Milyutin focused on pure mathematics. But he did not give up applied problems and continued to be involved in the statement and calculation of specific problems which were important for the Institute.'

One the most brilliant results of the late 1950s, the result which made a real stir and infused life into a new development in mathematics and mathematical applications was L.S. Pontryagin's famous maximum principle. The maximum principle proof obtained by L.S. Pontryagin together with his co-workers V.G. Boltyansky, R.V. Gamkrelidze and Y.F. Michshenko was rather different, in terms of style, than the proofs of extremum necessary conditions well known in analysis and calculus of variations. To some extent, it seemed natural due to the originality and a much higher degree of complexity of optimal control problems as compared to calculus of variations problems. Nevertheless, the issue of the connection between and continuity of calculus of variations and optimal control remained undetermined. This issue stimulated development of new proofs of the maximum principle and new approaches to obtaining necessary first-order conditions in optimal control in this country and abroad. For a while optimal control turned into a real Klondike where everyone tried to find a golden bullion.

The question about whether there was a way to prove the maximum principle using traditional approaches and methods was the one A.Y. Povzner asked of two ICP fellows, A.Y. Dubovitsky and A.A. Milyutin. Quite unexpectedly, the question became a turning-point for both of them and twisted their future work (which they did jointly for a rather long period of time) with optimal control. In 1965 the *Computing Mathematics and Mathematical Physics* journal published, on N.N. Moiseev's initiative, an article by A.Y. Dubovitsky and A.A. Milyutin, "Extremum Problems under Constraints". The article immediately won popularity and became pivotal for the authors and some of their followers due to its unique clarity and elegance.

Any necessary first-order condition of local minimum in a problem with constraints was interpreted in the article as a condition of nonintersection of the constraints approximation with the approximation of minimized functional decrease set; the set of decrease and all inequality constraints were equivalent and examined independently of each other. First-order approximations for inequalities (under natural assumptions) are open convex cones while equality constraints approximation is just an open cone (as a rule, it's a subspace according to L.A. Lyusternik's theorem on tangent manifold). The condition of nonintersection of cones, which the authors called the Eiler equation, is exactly the necessary first-order condition. When defined, the condition needs to be explained in the language of the field of mathematics where the problem is explored. In optimal control, the condition of cone nonintersection leads to the condition which is typically referred to as the Eiler-Lagrange condition, or according to the authors, the ‘local maximum principle”.

Another brilliant and somewhat unexpected idea described in the article was the idea of variation by way of replacement of the independent variable (time), or *v* -replacement, where *v* is a derivative of the function that makes the replacement. Any monotonous time replacement (i.e. replacement with the nonnegative function *v*) creates the so-called ‘attached problem’ and a new optimal trajectory in it. Small variations of function in the attached problem lead to nonsmall variations of control in the original problem, which act as Weierstrass needle-shaped variations used to obtain the maximum principle initially.

The local maximum principle described in each attached problem is rewritten as a ‘partial maximum principle’ in the original problem. The maximum principle results from the organization of partial maximum principles in the original problem (which was possible in all the cases explored at the time, and many other cases as well).

So the maximum principle received a new important interpretation: it turned out to be equivalent to the condition of stationarity in each attached problem (what is meant here is the stationarity of the trajectory which stemmed from the original trajectory by way of v-replacement). Later, this interpretation served as a reliable point of reference in cases where the maximum principle was extended to broader classes of problems.

Problems with state constraints were the first to be covered. The maximum principle for problems with state constraints was the subject of the doctorate thesis A.A. Milyutin defended brilliantly at the Institute of Applied Mathematics (USSR Academy of Sciences) in 1966. In addition to major findings related to the maximum principle the thesis explored an example of an extremal which, when ‘put’ on the boundary of a state constraint, has a countable number of contacts with the boundary on the finite segment of the time preceding the movement along the boundary. The control here has a countable number of switchings which accumulate towards the point (chattering). The moments of the switchings formed a decreasing geometric progression – that was related to the self-similarity of the trajectory. Later a similar example was independently discovered and computed by Robbins.

In the late 1960s and 1970s A.Y. Dubovitsky and A.A. Milyutin jointly developed the theory of the maximum principle for problems with regular and nonregular mixed constraints, which they published in a number of articles. Their extraordinary achievement was the local maximum principle for nonregular mixed constraints, explored in the so-called ‘White Book’, *Necessary Conditions of Extremum in the General Problem of Optimal Control,* 1971, Nauka Publishers, Moscow. Unfortunately with the circulation of just 500 copies, the book has long become a bibliographical piece. The book conducts a trenchant analysis of the Eiler equation for problems with mixed constraints, which involves functionals from the space conjugate to L_{∞} , in particular their singular constituents. The analysis made sure that no information was coarsened or lost. The answer was given in the terms of summed functions and Radon measures and by its characteristics it was close to the solution previously obtained for problems with state constraints.

The authors’ subsequent efforts were made towards obtaining an integral maximum principle for problems with nonregular mixed constraints. It turned out however that unlike in the case of regular problems, in the general case one can’t expect to obtain a single maximum principle, instead one has to deal with a whole hierarchy of maximum principles without the ‘maximum element’. So the authors focused on looking for the best possible way to present and organize the hierarchy. That was the subject of A.Y. Dubovitsky’s doctorate thesis. Later A.A. Milyutin found a new form of presenting the conditions of the maximum principle; the form reflected multiplicity and hierarchy of maximum principles in the general problem, as well as new ways of obtaining those. That material was presented in A.A. Milyutin’s monograph, *The Maximum Principle in the General Problem of Optimal Control*, 2001, PhizMatLit Publishers, Moscow.

Besides conducting research, in the late 1960s and early 1970s Milyutin started holding lectures and seminars for the DMM students at Moscow State University. It was at the seminar which Milyutin co-led with Y.S. Levitin, that he began in-depth research into the theory of higher order conditions. He raised the issue of obtaining necessary second order conditions in optimal control, which would be connected with the sufficient conditions as closely as in the problems of analysis and calculus of variations. The research resulted in the general theory of higher order conditions in problems with constraints, to which the new concept of condition order was central. Now the order was interpreted as nonnegative functional in variation space, which served as assessment for the problem’s functional increment at the point of minimum in admissible variations and determined the degree of the roughness of the condition in question. That theory was published in the article by Y.S Levitin, A.A. Milyutin, N.P. Osmolovsky in *Russian Mathematical Surveys,* #6, 1978 dedicated to L.S. Pontryagin’s 70th birthday. The theory offered landmark approaches to obtaining higher order conditions in optimal control, and the results were ready to follow. The first serious findings by the authors and A.V. Dmitruk were described in the above article; later A.V. Dmitruk and N.P. Osmolovsky, Milyutin’s former students, generalized their findings pertaining to the theory of quadratic conditions in optimal control for singular and nonsingular extremals respectively, and defended their doctorate theses on the subject. A.A. Milyutin led their research and took an active part in it. At about the same time he proved the remarkable ‘theorem on finite commensurability’ which revealed the true meaning of a whole number of findings by other mathematicians (A.A. Agrachev, R.V. Gamkrelidze, Krener and others) in the field of higher order necessary conditions for singular regimes in optimal control.

In those years, mathematicians who focused on the extremum theory, including A.D. Ioffe, V.M. Tikhomirov, V.F. Sukhinin and others, were discovering new forms of the theory’s central piece, Lyusternik’s theorem on tangent manifold. A.A. Milyutin offered a different interpretation of Lyusternik’s theorem and defined it as a ‘theorem on covering’. More interpretations were found later. The review of the findings was published by A.V. Dmitruk, A.A. Milyutin and N.P. Osmolovsky in the *Russian Math Surveys* issue dedicated to L.A. Lyusternik’s 80th birthday. But the theorem on covering turned out to be the easiest and most understandable in terms of its formulation and very practical in terms of its use. That’s the reason why it’s gaining popularity, in nonsmooth analysis among other fields (see A.D. Ioffe’s article in *Russian Math Surveys,* Vol. 55, Issue 3, 2000).

In about mid-80s Milyutin changed focus from obtaining new extremum conditions to making these conditions as much suitable for investigating new phenomena in optimal control as possible. That led to the theorems on the special structure of Lagrangian multipliers in the maximum principle conditions (theorems on the absence of jumps and theorems on the absence of singular components in Lagrangian multiplier measures under state constraints), which were described in the monograph *Necessary Condition in Optimal Control*, 1990, Nauka Publishers, Moscow. Moreover, Milyutin was thoroughly exploring the phenomena in optimal control and other fields of mathematics using the tools of the maximum principle and higher order conditions.

The maximum principle was used to investigate extremals when they are put on the boundary of a state constraint and get off it. The findings were described in the monograph by V.V. Dikusar and A.A. Milyutin *Qualitative and Numerical Methods in Maximum Principle*, 1989, Nauka Publishers, Moscow. Later, A.A. Milyutin developed an exhaustive solution of a number of special problems related to putting extremals on the state constraint. There he defined the conditions under which the extremal putting was accompanied by a countable number of contacts with the state constraint.

Simultaneously, the maximum principle was used to investigate the behavior of extremals during their passage through a singular manifold (in particular, during the landing on a singular extremal), the nonuniqueness funnel for extremals, the conditions for the chattering-type effect or, more generally, discontinuity of the second kind of control. S.V. Chukanov took an active part in the research. The results were published in *Optimal Control in Linear Systems* by A.A. Milyutin, A.E. Ilyutovich, N.P. Osmolovsky, and S.V. Chukanov.

A.A. Milyutin used quadratic conditions to investigate the concept of rigidity in optimal control. The complete system of quadratic conditions is transferred to rigidity. That results in obtaining characteristics of sets of quadratically rigid trajectories in contact structures for the arbitrary dimension space and determining the structure and dimension of the quadratically rigid trajectory set in a contact structure depending on the space dimension. Milyutin built a seemingly surprising example of a quadratically rigid trajectory with a nonunique set of normed Lagrangian multipliers where the quadratic form corresponding to each set is not even negatively defined in a set of critical variations (the example is no longer surprising when one gets to know the complete system of quadratic conditions for singular extremals obtained by A.V. Dmitruk).

Together with A.V. Dmitruk, A.A. Milyutin used quadratic conditions to analyze singular geodesics relative to submetrics. In particular, he established nonuniformity (in terms of extremal characteristics) of the vicinity structure of a singular geodesic relative to a submetric and proved that in the topology of uniform convergence of functions with their derivatives, each locally quadratically rigid geodesic is ultimate for the sequences of two types of nonsingular extremals: the sequences where all couples of conjugate points converge at an arbitrary small distance, and the sequences which do not have subsequences of converging couples of conjugate points.

The problems arising in the quadratic theory of singular extremals triggered research into the possibility of approximation of random vector fields in a finite-dimensional space with gradient fields. Milyutin found a duality formula which connects normed vector field circulation with a distance (in a sense) between that field and a set of gradient vector fields. Later, the formula was effectively used in the quadratic theory of singular extremals.

Together with V.L. Bodneva, A.A. Milyutin generalized the Bogolyubov-Krylov asymptotic method for the case of the differential equation right side's random dependence on the parameter. The parameter here may be an element of the metric space. Recursions of the generalized asymptotic process were obtained (Russian Math Surveys, 1987, vol.42, N 3). That approach helped the authors (Milyutin and Bodneva) obtain new interesting results related to the mathematical theory of vibrations, where vibration, a sequence weakly L_{∞} converging to zero, is viewed as a small parameter. (Russian J. Of Math. Phys., 1998, vol.5, N2). The proposed method goes beyond the known averaging methods and helps obtain ultimate systems in the case of right sides which are continuous in a state constraint and in the case of discontinuous right sides as well (an example being problems involving dry friction).

Together with N.P. Osmolovsky, A.A. Milyutin explored the movement and interpenetration of ideas between calculus of variations and optimal control. Milyutin started this exploration with a course of lectures on calculus of variations at the DMM of Moscow State University. By analogy with the weak minimum theory which makes the foundation of the classical calculus of variations, what is built is the calculus of the so-called Pontryagin minimum explored in the optimal control theory. The Pontryagin minimum theory was described in a monograph by A.A. Milyutin and N.P. Osmolovsky, *Calculus of Variations and Optimal Control*, published by the American Mathematical Society in their Translations of Mathematical Monographs series in 1998.

After his trip to Israel in 1998 where he attended a conference dedicated to the tercentenary of calculus of variations, Milyutin focused on the findings in the field of the maximum principle theory for differential inclusions. He explored the influence of the Holder condition for a differential inclusion on the form of the maximum principle and established that for a differential inclusion whose right side is prescribed by convex-valued mapping which meets the Holder condition, in the absence of state and mixed constraints there is no maximum principle with continuous conjugate variables (which gives a partial answer to Clarke's similar question about the Lipschitz inclusions). For differential inclusions, Milyutin also established the nonequivalence of the maximum principle to the Ecklund principle. Using the maximum principle in optimization problems for differential inclusions he identified finer necessary conditions that those obtained earlier by Clarke who used the Ecklund principle in the same problems. He also used the maximum principle to enhance the Smirnov conditions.

Unfortunately this is too short a review to mention all important and numerous results obtained by A.A. Milyutin.

I.L. Barsky, V.L. Bodneva, A.V. Dmitruk, M.I. Zelikin, Y.M. Kazhdan, S.L. Kamenomostskaya, A.M. Kogan, A.M. Molchanov, N.P. Osmolovsky, V.M. Tikhomirov, A.V. Fursikov, E.E. Shnol