An Example of an Optimal Control Problem Whose Extremals Possess a Continual Set of Discontinuities of the Control Function

ALEXEI A. MILYUTIN

Institute of Chemical Physics of the Russian Academy of Sciences, 2 ul. Kosygina, Moscow, Russia

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We present an example of a classical optimal control problem having solutions that are nonsingular extremals and whose control functions have a continual discontinuity set. Moreover, for any closed, nowhere-dense set, there exists a solution such that the set of discontinuities of the control coincides with this set.

We consider the control system

$$\dot{z} = (\mathcal{E}yy, u), \quad \dot{x} = |y|^2, \quad \dot{y} = u, \quad |u| \leqslant 1 \tag{1}$$

and the optimization problem

$$y = \frac{z(T) - z(0)}{x(T) - x(0)} \to \max \qquad \text{under the constraint} \quad y(0) = y(T). \tag{2}$$

Here $\mathcal{E}yy = (\mathcal{E}_1 yy, \mathcal{E}_2 yy)$ is a quadratic vector field on the two-dimensional y-plane.

This problem belongs to a class of problems which is important for higher order condition theory. However, the problem itself is of interest since it extends our comprehension of singularities in control systems.

We assume that the field $\mathcal{E}yy$ is not a gradient one, since otherwise the functional J vanishes on any admissible trajectory. Setting

$$l(y) = \frac{\partial \mathcal{E}_1 yy}{\partial y_2} - \frac{\partial \mathcal{E}_2 yy}{\partial y_1},$$

we readily see that $l(y)$ is a linear function of y. Therefore $l(y) = (l, y)$, where l is some vector from the plane $\mathbb{R}^2$. Our assumption means that $l \neq 0$. We shall find all solutions of the problem (1), (2). Let us write out the most important conditions.

The Pontryagin function of the problem has the form: $H = \psi_z(\mathcal{E}yy, u) + \psi_x|y|^2 + \psi_y u$. Since H does not depend explicitly on z, x, and t, it follows from the adjoint equations that

$$\psi_z = \text{const}, \qquad \psi_x = \text{const}, \qquad H = \text{const}$$

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on any extremal of the system (1). The transversality conditions give:

$$\psi_z = \frac{\alpha}{x(T)-x(0)}, \quad \psi_x = -\alpha\frac{z(T)-z(0)}{(x(T)-x(0))^2}, \quad \alpha \geqslant 0$$
$$\psi_y(0) = \psi_y(T) = \beta, \quad H = 0. \tag{3}$$

According to the nontriviality condition, we have $\alpha + |\beta| > 0$. Let us write out the Hamiltonian:

$$\mathcal{H} = \max_u H = \left|\psi_z \mathcal{E}yy + \psi_y\right| + \psi_x |y|^2.$$

It follows from condition (3) that the solution of the problem (more precisely, the corresponding extremal) lies on the zero level of the Hamiltonian. Now we can easily prove that $\alpha > 0$. In fact, otherwise we would have $\psi_z = \psi_x = 0$. Then from $\mathcal{H} = 0$ it follows that $\psi_y = 0$, hence $\beta = 0$, and the nontriviality condition is violated. Thus, $\alpha > 0$. Taking into account the obvious inequalities

$$z(T) - z(0) > 0, \quad x(T) - x(0) > 0,$$

we obtain $\psi_z > 0, \psi_x < 0$.

In order to solve problem (1), (2), we use the Hamilton-Jacobi equation which has the form

$$\left|\frac{\partial S}{\partial z}\mathcal{E}yy + \frac{\partial S}{\partial y}\right| + \frac{\partial S}{\partial x}|y|^2 + \frac{\partial S}{\partial t} = 0. \tag{4}$$

On account of the transversality condition, it is natural to suppose that $S = z - \gamma x + s(y)$. Then the Hamilton-Jacobi equation takes the form:

$$|\mathcal{E}yy + s'_y| - \gamma|y|^2 = 0. \tag{5}$$

Since s'_y is related, by virtue of this equation, to the quadratic field $\mathcal{E}yy$, one may guess that s'_y is also a quadratic field, i.e., the function $s(y)$ is a cubic form. Put $\widehat{\mathcal{E}}yy = \mathcal{E}yy + s'_y$. Then it follows from (5) that

$$|\widehat{\mathcal{E}}yy| = \gamma, \quad |y| = 1, \quad \frac{\partial\widehat{\mathcal{E}}_1yy}{\partial y_2} - \frac{\partial\widehat{\mathcal{E}}_2yy}{\partial y_1} = (l, y). \tag{6}$$

Let us find all solutions of (6). Let

$$\widehat{\mathcal{E}}_1yy = a_1y_1^2 + 2b_1y_1y_2 + c_1y_2^2, \quad \widehat{\mathcal{E}}_2yy = a_2y_1^2 + 2b_2y_1y_2 + c_2y_2^2.$$

Put $y_1 = \cos\varphi, y_2 = \sin\varphi$, then we obtain:

$$\widehat{\mathcal{E}}_1 y(\varphi)y(\varphi) = r_1(\varphi) = A_1\cos 2\varphi + B_1\sin 2\varphi + C_1,$$

where $A_1 = (a_1 - c_1)/2$, $B_1 = b_1$, $C_1 = (a_1 + c_1)/2$. Similarly, $r_2(\varphi) = A_2\cos^2\varphi + B_2\sin 2\varphi + C_2$, where

$$A_2 = \frac{a_2 - c_2}{2}, \qquad B_2 = b_2, \qquad C_2 = \frac{a_2 + c_2}{2}. \tag{7}$$

If we set $r(\varphi) = (r_1(\varphi), r_2(\varphi))$, then (7) implies that the condition $|r(\varphi)| = \text{const}$ is equivalent to one of the following two conditions:

$$a) A_1 = B_1 = A_2 = B_2 = 0,$$
$$b) A_1^2 + B_1^2 = A_2^2 + B_2^2, \quad A_1B_1 + A_2B_2 = 0, \quad C_1 = C_2 = 0. \tag{8}$$

Put $l = (l_1, l_2)$, then, in view of (7), we have

$$l_1 = 2(B_1 - A_2 - C_2), \quad l_2 = 2(-B_2 - A_1 + C_1). \tag{9}$$

In case a), we have $C_1 = l_2/2$, $C_2 = -l_1/2$. Therefore,

$$\widehat{\mathcal{E}}yy = \left(\frac{l_2}{2}(y, y), -\frac{l_1}{2}(y, y)\right), \quad \gamma = \frac{|l|}{2}. \tag{10}$$

In case b), we have in view of (9):

$$\frac{l_1}{2} = B_1 - A_2, \qquad \frac{l_2}{2} = -B_2 - A_1.$$

However, it follows from (7) that

$$(B_1, B_2) = \pm(-A_2, A_1).$$

It is clear that only the first possibility is to be retained, since in the second case $l_1 = l_2 = 0$. Thus, $B_1 = -A_2, B_2 = A_1$. Now we can express $A_1, A_2; B_1, B_2$ by means of l_1 and l_2. We obtain

$$A_1 = -\frac{l_2}{4}, \quad B_1 = \frac{l_1}{4}, \quad A_2 = -\frac{l_1}{4}, \quad B_2 = -\frac{l_2}{4}.$$

Therefore, in the case b) the following equalities hold:

$$\widehat{\mathcal{E}}_1 yy = -\frac{l_2}{4}(y_1^2 - y_2^2) + \frac{l_1}{2} y_1 y_2, \quad \widehat{\mathcal{E}}_2 yy = -\frac{l_1}{4}(y_1^2 - y_2^2) - \frac{l_2}{2} y_1 y_2, \quad \gamma = \frac{|l|}{4}. \tag{11}$$

Equalities (10) and (11) contain the complete solution of equations (6). We consider the cases (10) and (11) simultaneously. Put

$$u(y) = \frac{\widehat{\mathcal{E}}yy}{|\widehat{\mathcal{E}}yy|}.$$

The function $u(y)$ is well-defined everywhere excluding the point $y = 0$, since in both cases (10) and (11) we have $\gamma > 0$. It is obvious that $u(y)$ is smooth for $y \neq 0$ and

$$|u(y)| = 1 \quad \forall y \in \mathbb{R}^2 - \{0\}.$$

Let $z(t)$, $x(t)$, $y(t)$ be an arbitrary solution defined in some interval $[t_0, t_1]$ and contained in the domain $y \neq 0$ of the system

$$\dot{z} = (\mathcal{E}yy, u(y)), \quad x = |y|^2, \quad \dot{y} = u(y). \tag{12}$$

Let $S = z - \gamma x + s(y)$ be a solution of equation (5). Then it is easy to see that

$$S = \text{const} \tag{13}$$

along the above trajectory. In fact,

$$\begin{aligned}\frac{dS}{dt} &= (\mathcal{E}y(t)y(t), u(y(t))) - \gamma|y(t)|^2 + \frac{\partial S}{\partial y} u(y(t)) \\ &= (\widehat{\mathcal{E}}y(t)y(t), u(y(t))) - \gamma|y(t)|^2 = |\widehat{\mathcal{E}}y(t)y(t)| - \gamma|y(t)|^2 = 0.\end{aligned}$$

Therefore our assertion is true. On the other hand, if $z(t), x(t), y(t), u(t)$ is an arbitrary trajectory of the system (1) such that $y(t) \neq 0$ for all t in some interval $[t_0, t_1]$ and the inequality $u(t) \neq u(y(t))$ holds on a subset of positive Lebesgue measure, then

$$S(z(t_1), x(t_1), y(t_1)) - S(z(t_0), x(t_0), y(t_0)) < 0. \tag{14}$$

The proof of this assertion is similar to the proof of assertion (13). In fact, for such a trajectory

$$\frac{dS}{dt} = (\widehat{\mathcal{E}}y(t)y(t), u(t)) - \gamma|y(t)|^2 \leqslant |\widehat{\mathcal{E}}y(t)y(t)| - \gamma|y(t)|^2 = 0,$$

and the inequality is strict if $u(t) \neq u(y(t))$. Now, (14) follows easily.

Now we again consider the cases (10) and (11) separately. In case (10),

$$u(y) = \frac{Vl}{|l|}, \quad V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and, consequently, there are no cycles among solutions of the equation $\dot{y} = u(y)$. Therefore, there are no admissible trajectories of the problem (1),(2) among solutions of system (12). Thus, case (10) has nothing to do with the solution of problem (1), (2). In case (11), we represent $u(y)$ in the form:

$$u(y) = (u_1(y), u_2(y)),$$
$$u_1(y) = \frac{1}{|y|^2}\left(-\frac{l_2}{|l|}(y_1^2 - y_2^2) + \frac{l_1}{|l|}2y_1y_2 \right),$$
$$u_2(y) = \frac{1}{|y|^2}\left(-\frac{l_1}{|l|}(y_1^2 - y_2^2) - \frac{l_2}{|l|}2y_1y_2 \right).$$

Put $\zeta = y_1 + iy_2, \theta = -l_2/|l| - il_1/|l|$. Then

$$u_1(y) = \frac{1}{|\zeta|^2}\Re(\theta\zeta^2), \qquad u_2(y) = \frac{1}{|\zeta|^2}\Im(\theta\zeta^2).$$

Therefore, the last equation of system (12) can be rewritten in the form

$$\dot{\zeta} = \frac{1}{|\zeta|^2}\theta\zeta^2. \tag{15}$$

Put $d\tau = dt/|\zeta|^2$. Then equation (15) takes the form $d\zeta/d\tau = \theta\zeta^2$. Up to a shift in τ, the general solution of this equation is given by

$$\zeta = \frac{1}{\theta}\ \frac{1}{\tau + iC}, \tag{16}$$

where C is an arbitrary real number. For $C = 0$ we do not get a cyclic (periodic) solution of equation (15). However, for any $C \neq 0$ the corresponding solution of (15) will be cyclic. We put

$$t = \int_{-\infty}^{\tau} \frac{d\tau}{\tau^2 + C^2} \quad \text{for} \quad C \neq 0.$$

Then the corresponding solution $y(t,C)$ of equation (12) takes the form:

$$y(t,C)=\left(\Re\left(\frac{1}{\theta}\frac{1}{\tau(t)+iC}\right),\quad \Im\left(\frac{1}{\theta}\frac{1}{\tau(t)+iC}\right)\right).$$

It is obvious that $y(t,C)$ is a solution of (12) in the closed interval $[0,T(C)]$ where $T(C)=\pi/|C|$. The solution $y(t,C)$ is continuous in the closed interval $[0,T(C)]$, does not vanish in the open interval $(0,T(C))$, and vanishes at its endpoints. Therefore, the function $y(t,C)$ is an admissible trajectory for any $C\neq 0$.

By using (13) and (14), it is not hard to show that $y(t,C)$ is a solution of problem (1), (2) for any $C\neq 0$. In fact, let $z(t),x(t),y(t,C)$ be a solution of system (12). Then, according to (13),

$$z(T(C))-z(0)-\frac{|l|}{4}(x(T(C))-x(0))=0.$$

Hence,

$$\frac{z(T(C))-z(0)}{x(T(C))-x(0)}=\frac{|l|}{4}. \tag{17}$$

Now let $z(t),x(t),y(t),u(t)$ be any admissible trajectory defined in a closed interval $[0,T]$. Then, in view of (14),

$$z(T)-z(0)-\frac{|l|}{4}(x(T)-x(0))\leqslant 0.$$

If $y(t)\neq 0$, i.e., if $x(T)-x(0)>0$, then it follows that

$$\frac{z(T)-z(0)}{x(T)-x(0)}\leqslant\frac{|l|}{4}. \tag{18}$$

The comparison of (17) and (18) gives the desired conclusion.

The existance of a multiplicity of solutions is related to the invariance of problem (1), (2) under a similarity group.

Namely, if $y(t)$ is an admissible function, then the function $\lambda y(t/\mu)$, $|\lambda/\mu|=1$, is also admissible and gives the same value of the functional J as the initial one. By using the solutions $y(t,C),C\neq 0$ and the fact that problem is invariant with respect to shifts of the variable t, we shall construct the general solution of the problem. Let $[0,T]$, $T>0$, be an arbitrary interval, and $\mathcal{F}\subset[0,T]$ be a closed set which contains the endpoints of the interval but does not coincide with it. We define a function $y_{\mathcal{F}}(t)$. First we set

$$y_{\mathcal{F}}(t)=0\quad\forall t\in\mathcal{F}.$$

Let $\Delta\subset[0,T]$ be the complementary interval of $\mathcal{F}$. Put

$$y_{\mathcal{F}}(t)|_{\Delta}=y(t-t_L(\Delta),C(\Delta)),$$

where $t_L(\Delta)$ is the left end of Δ, $|C(\Delta)|=\pi|\Delta|$. Now the function $y_{\mathcal{F}}(t)$ is completely defined. It is easy to see that $y_{\mathcal{F}}(t)$ satisfies the Lipschitz condition with constant 1 in $[0,T]$ and that $y_{\mathcal{F}}(0)=y_{\mathcal{F}}(T)=0$. Thus, $y_{\mathcal{F}}(t)$ is an admissible function. Since $\mathcal{F}$ does not coincide with the entire interval $[0,T]$, this function is not equal to zero on a set of positive Lebesgue measure.

Put $u_{\mathcal{F}}(t)=\frac{d}{dt}y_{\mathcal{F}}(t)$. It is obvious that

$$\frac{\int_0^T(\mathcal{E}y_{\mathcal{F}}y_{\mathcal{F}},u_{\mathcal{F}})\,dt}{\int_0^T|y_{\mathcal{F}}|^2dt}=\frac{\sum_\Delta\int_\Delta(\mathcal{E}y_{\mathcal{F}}y_{\mathcal{F}},u_{\mathcal{F}})dt}{\sum_\Delta\int_\Delta|y_{\mathcal{F}}|^2\,dt}.$$

But

$$\frac{\int_\Delta (\mathcal{E}y_{\mathcal{F}}y_{\mathcal{F}}, u_{\mathcal{F}})\, dt}{\int_\Delta |y_{\mathcal{F}}|^2 dt} = \frac{|l|}{4} \quad \forall\Delta.$$

Consequently, $y_{\mathcal{F}}(t)$ is a solution of problem (1), (2).

The family $y_{\mathcal{F}}(t)$ contains essentially all the solutions of this problem. Let $y(t)$ be a solution of the problem in a certain interval $[0, T]$. It is not hard to show that there exists a point t in the interval such that $y(t) = 0$. Otherwise, according to (14), the equation $\dot{y} = u(y)$ must hold on the entire segment. However, as we have already seen, it does not possess a cyclic solution in the domain $y \neq 0$. Let $t_0 \in [0, T]$, $y(t_0) = 0$. By a change of variables we reduce the situation to the case $t_0 = 0$. To this end we put:

$$y_1(t) = \begin{cases} y(t_0 + t), & t_0 + t \leqslant T, \\ y(t_0 + t - T), & t_0 + t > T. \end{cases}$$

Then it is obvious that $y_1(t)$ is an admissible trajectory in $[0, T]$ and $y_1(0) = y_1(T) = 0$. It is quite clear that $J[y_1(\cdot)] = J[y(\cdot)]$, and thus $y_1(t)$ is also a solution.

Denote by $\mathcal{F}$ the set of zeros of the function $y_1(t)$ in the interval $[0, T]$. It is clear that $\mathcal{F}$ satisfies all the necessary conditions. We shall prove that

$$y_1(t) = y_{\mathcal{F}}(t). \tag{19}$$

Let Δ be the complementary interval of $\mathcal{F}$. Then $y_1(t)$ is a cyclic solution of the equation $\dot{y} = u(y)$ in Δ. Hence, there exist C, $|C| = \pi/|\Delta|$, such that

$$y_1(t)|_\Delta = y(t - t_L(\Delta), C).$$

Eq. (19) is now proved.

Hence, an arbitrary solution of the problem differs from the corresponding solution of the family $\{y_{\mathcal{F}}(t)\}$ only by the choice of the initial point of the cycle. Thus, the problem (1),(2) is completely solved

We note the following fact. Each solution of problem (1), (2) is a trajectory component of an extremal. The adjoint variables of any solution are described, according to the Hamilton-Jacobi theory, as follows:

$$\psi_z = 1, \quad \psi_x = -\frac{|l|}{4}, \quad \psi_y = s'_y(y(t)),$$

where $s'_y(y) = \widehat{\mathcal{E}}yy - \mathcal{E}yy$, $y(t)$ is a component of the solution. The extremal is singular on the set of points t where $\widehat{\mathcal{E}}yy = 0$ if the measure of this set is positive. It follows from (6) that this set coincides with the zero set of $y(t)$. On the complementary set $y(t) \neq 0$ (or, what is the same, $\widehat{\mathcal{E}}yy \neq 0$) the extremal is nonsingular since the control is expressed, according to the maximum principle by means of y and ψ_y.

Let $\mathcal{F}$ be a perfect set and $\operatorname{mes}\mathcal{F} = 0$. Then, since $\mathcal{F}$ is just the zero set of $y_{\mathcal{F}}(t)$, the corresponding extremal is nonsingular. On the other hand, since there are infinitely complementary intervals in any neighborhood of any point of $\mathcal{F}$, any point of $\mathcal{F}$ is a point of discontinuity of the control $u_{\mathcal{F}}(t)$. Therefore, the system (1) is an example of a control system having nonsingular extremals such that the control has a continuum of discontinuities.

In the theory of higher order conditions for singular extremals due to A. V. Dmitruk, a somewhat more general problem than (1), (2) is important. Namely, instead of the constraint $|u| \leqslant 1$, one must consider the restriction $u \in U$, where U is a convex compact set such that $0 \in \operatorname{int} U$. Using the above ideas, A. V. Dmitruk shows that if the boundary of U is a shifted ellipse, then solutions of this new problem have a structure similar to that of problem (1), (2). This result is unpublished. It seems that this result holds in dimension 2 for any U, while the proof requires much more complicated considerations.

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