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An Example of a Rigid Trajectory of a Two-Dimensional Distribution Such That, for Any Associated Extremal, the Conjugate Point Is Inside the Corresponding Closed Interval*

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Abstract. We construct an example of a rigid trajectory of a two-dimensional distribution such that, for any associated extremal, the conjugate point belongs to the interior of the corresponding closed interval.

INTRODUCTION

We base our research upon [1]. In our opinion, the sufficient rigidity conditions presented there are more complete as compared with those previously known, and the form of the conditions in [1] is more convenient in applications.

For the reader's convenience, we preface the example by some general results from [1] that are stated here without proof.

In the space $\mathbb{R}^{d(x)}$ of elements $x = (x_1, \dots, x_{d(x)})$, where $d(x)$ is the dimension of the space, we consider a generalized differential equation of the form

$$\dot{x} \in \Gamma(x), \tag{1}$$

where $\Gamma(x)$ is an m-dimensional subspace of $\mathbb{R}^{d(x)}$, defined in some domain. We assume that in this domain there is a basis $\Gamma(x) = \{\mathbf{r}_0(x), \dots, \mathbf{r}_{m-1}(x)\}$ whose elements are twice continuously differentiable with respect to x. A trajectory of the generalized differential equation (1) is defined as a function $x(t)$ defined on an interval $[t_0, t_1]$ and satisfying both the Lipschitz condition on this interval and the generalized differential equation (1) almost everywhere. Since we are interested in the rigidity condition, it follows that the generalized differential equation (1) seems to be most convenient because, on one hand, it needs no formulations in invariant form and, on the other hand, leads to no loss of generality.

A trajectory $x^0(t) \mid [t_0, t_1]$ is said to be *rigid* if, for any sequence of trajectories of the generalized differential equation (1), say, $\{x^s(t) \mid [t_0, t_1]\}$, such that $x^s(t_0) = x^0(t_0)$, $x^s(t_1) = x^0(t_1)$, and

$$v \max_{[t_0, t_1]} |\dot{x}^s(t) - \dot{x}^0(t)| \to 0 \quad \text{as} \quad s \to \infty,$$

there exists (starting from some index $s = s_*$) a sequence of strictly increasing functions $\varphi^s(t)$ such that $\varphi^s(t_0) = t_0$, $\varphi^s(t_1) = t_1$, and $x^s(t) = x^0(\varphi^s(t))$ for any $t \in [t_0, t_1]$.

Each rigid trajectory has a nonempty set of associated extremals.

We define extremals as follows. Let $(\mathbf{r}_0(x), \dots, \mathbf{r}_{m-1}(x)) = \widehat{\mathbf{b}}$ be a basis of the distribution $\Gamma(x)$. For $\mathbf{r} \in \Gamma(x)$ we define a mapping $h(x, \mathbf{r}, \widehat{\mathbf{b}}) = \big(h_0(x, \mathbf{r}, \widehat{\mathbf{b}}), \dots, h_{m-1}(x, \mathbf{r}, \widehat{\mathbf{b}})\big)$ by the condition

$$\mathbf{r} = \sum_{i=0}^{m-1} h_i(x, \mathbf{r}, \widehat{\mathbf{b}}) \mathbf{r}_i(x).$$

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Let a trajectory $x(t) \mid [t_0, t_1]$ of the generalized differential equation (1) be given. We say that a pair $\psi(t), x(t) \mid [t_0, t_1]$, where $\psi(t)$ is an absolutely continuous function on $[t_0, t_1]$ is an *associated extremal* for the trajectory $x(t) \mid [t_0, t_1]$ if the following conditions hold:

$$\begin{aligned} -\dot{\psi}(t) &= \psi(t) \sum_{i=0}^{m-1} h_i(x(t), \dot{x}(t), \widehat{\mathbf{b}}) \mathbf{r}_i'(x(t)), \\ \psi(t)\mathbf{r}_i(x(t)) &= 0 \qquad \forall t \in [t_0, t_1],\ i = 0, \dots, m-1; \\ \psi(t)[\mathbf{r}_i, \mathbf{r}_k](x(t)) &= 0 \qquad \forall t \in [t_0, t_1],\ i, k = 0, \dots, m-1, \qquad \psi(t) \not\equiv 0, \end{aligned} \tag{2}$$

where we use the notation $[\rho_1, \rho_2](x) = \rho_1'\rho_2 - \rho_2'\rho_1$, and $\mathbf{r}'$ stands for $\mathbf{r}_x'$. In the definition of the corresponding extremal, the above basis $\widehat{\mathbf{b}}$ is used; however, this definition does not depend on the choice of $\widehat{\mathbf{b}}$.

A function $\psi(t)$ is said to be a *conjugate component* of the extremal $\psi(t), x(t) \mid [t_0, t_1]$. We denote the set of conjugate components of the associated extremals normalized by some condition by $\Psi_0(\tau)$, where $\tau = x(t) \mid [t_0, t_1]$ is a given trajectory.

Concerning a trajectory whose rigidity is studied, we assume that

$$\begin{gathered} \dot{x}(t) \quad \text{is twice continuously differentiable on} \quad [t_0, t_1]; \\ \dot{x}(t) \neq 0 \quad \text{for any} \quad t \in [t_0, t_1]; \qquad t', t'' \in [t_0, t_1], \quad t' \neq t'' \implies x(t') \neq x(t''). \end{gathered} \tag{3}$$

A basis $\widehat{\mathbf{b}}$ is said to be *associated* to a trajectory $x(t)$ satisfying (3) if $\mathbf{r}_0(x(t)) = \dot{x}(t)$ for all $t \in [t_0, t_1]$. It is clear that for an adjoint basis we have

$$h_0(x(t), \dot{x}(t), \widehat{\mathbf{b}}) = 1, \quad h_i(x(t), \dot{x}(t), \widehat{\mathbf{b}}) = 0, \quad i = 1, \dots, m-1.$$

Assume that a trajectory $x^0(t) \mid [t_0, t_1]$ satisfies (3) and that $\widehat{\mathbf{b}}$ is an associated basis. Let us consider the problem

$$\begin{gathered} J = \int_{t_0}^{t_1} y^2(t)\, dt \longrightarrow \max, \\ \dot{y} = u, \quad \dot{z} = 0, \quad \dot{x} = z\mathbf{r}_0(x) + \sum_{i=1}^{m-1} u_i \mathbf{r}_i(x), \\ y(t_0) = 0; \quad x(t_0) = x^0(t_0), \quad x(t_1) = x^0(t_1). \end{gathered} \tag{4}$$

Here we use the notation $y = (y_1, \dots, y_{m-1})$ and $u = (u_1, \dots, u_{m-1})$. Further we set $y^0(t) = 0$, $u^0(t) = 0 \mid [t_0, t_1]$, and $z^0 = 1$; then the formula

$$\widehat{\tau}^{\,0} = (y^0(t), z^0, x^0(t), u^0(t) \mid [t_0, t_1])$$

defines an admissible trajectory of problem (4). In [1], the following assertion is proved.

Theorem. *A trajectory $\tau^0 = x^0(t) \mid [t_0, t_1]$ is rigid if and only if $\widehat{\tau}^{\,0}$ realizes a weak extremum in problem* (4).

Therefore, the condition $\Psi_0(\tau^0) \neq \varnothing$ is necessary for rigidity. Subsequent conditions can be obtained from the theory of extrema [1] for problem (4). Let us formulate the general notion of quadratic rigidity.

By $\pi = \{\widehat{\tau}^s\}_{s=1,2\dots}$ we denote a sequence of trajectories of system (4) of the form

$$\widehat{\tau}^s = y^s(t), z^s, x^s(t), u^s(t) \mid [t_0, t_1]$$

such that $y^s(t_0) = 0$, $x^s(t_0) = x^0(t_0)$, $z_s \underset{s\to\infty}{\longrightarrow} 1$, and for which $\max_{[t_0,t_1]} |u^s(t)| \underset{s\to\infty}{\longrightarrow} 0$. The set of all such sequences π is denoted by Π_0. We assume that a sequence π is such that

$$(z^s - 1)^2 + \int_{t_0}^{t_1} (y^s)^2(t)\, dt > 0 \qquad \forall s.$$

The general notion of quadratic rigidity is in the following condition:

$$\inf_{\Pi_0} \lim_{\Gamma} \mid x^s(t_1) - x^0(t_1) \mid \Big/ \left((z^s - 1)^2 + \int_{t_0}^{t_1} (y^s)^2(t)\, dt + (y^s)^2(t_1) \right) > 0. \tag{5}$$

Formally, the basis $\widehat{\mathbf{b}}$ appears in condition (5). However, in the theory of extremum, it is proved that condition (5) does not depend on the choice of the associated basis. Now let us present an equivalent form of condition (5). The equivalent conditions are formulated for an associated basis $\widehat{\mathbf{b}}$.

Let $\psi(t) \in \Psi_0(\tau^0)$. Set $P(t) = \{P_{ik}(t)\}$, $i, k \in \{1, \dots, m-1\}$, where $P_{ik} = \psi(t)[[\mathbf{r}_0, \mathbf{r}_i], \mathbf{r}_k](x^0(t))$. For $\psi(\cdot) \in \Psi_0(\tau^0)$, the matrix $P(t)$ is symmetric. We set

$$\psi(\cdot) \in \mathcal{L}eg(\tau^0) \iff P(t) \geqslant 0 \quad \forall t \in [t_0, t_1]$$

(by the inequality $P(t) \geqslant 0$ we mean that the matrix $P(t)$ is nonnegative definite). The condition $\mathcal{L}eg(\tau^0) \neq \varnothing$ is also a necessary condition for the trajectory τ^0 be rigid.

Let us denote by $\overline{w}$ a collection $\overline{w} = (\overline{\beta}, \overline{z}, \overline{\xi}(t), \overline{y}(t))$, where $\overline{\beta} = (\overline{\beta}_1, \dots, \overline{\beta}_{m-1})$ is a constant vector, $\overline{z}$ is a constant, $\overline{\xi}(t) = (\overline{\xi_1}(t), \dots, \overline{\xi}_{d(x)}(t))$ is an absolutely continuous function on $[t_0, t_1]$, and $\overline{y}(t) = (\overline{y}_1(t), \dots, \overline{y}_{m-1}(t))$ is a measurable square-integrable function on $[t_0, t_1]$. Denote the set of all such functions $\overline{w}$ by $\overline{W}$. For $\psi(\cdot) \in \Psi_0(\tau^0)$ and $\overline{w} \in \overline{W}$, we define the form

$$\begin{aligned}
\omega(\psi(\cdot); \overline{w}) &= \omega_1(\psi(\cdot); \overline{w}) + \int_{t_0}^{t_1} \omega_2(t, \psi(\cdot), \overline{w})\, dt, \\
\omega_1(\psi(\cdot); \overline{w}) &:= \sum_{i=1}^{m-1} \overline{\beta}_i \psi(t_1) \mathbf{r}'_i(x^0(t_1)) \overline{\xi}(t_1) + \frac{1}{2} \sum_{i,k=1}^{m-1} \overline{\beta}_i \overline{\beta}_k \psi(t_1) \mathbf{r}'_i(x^0(t_1)) \mathbf{r}_k(x^0(t_1)). \\
\omega_2(t, \psi(\cdot); \overline{w}) &:= \frac{1}{2} \psi(t) \mathbf{r}''_0(x^0(t)) \overline{\xi}(t) \overline{\xi}(t) + \overline{z} \psi(t) \mathbf{r}'_0(x^0(t)) \overline{\xi}(t) \\
&\quad + \sum_{i=1}^{m-1} \overline{y}_i(t) \psi(t) [\mathbf{r}_0, \mathbf{r}_i]'(x^0(t)) \overline{\xi}(t) + \frac{1}{2} \sum_{i,k=1}^{m-1} \overline{y}_i \overline{y}_k \psi(t) [[\mathbf{r}_0, \mathbf{r}_i], \mathbf{r}_k](x^0(t)).
\end{aligned} \tag{6}$$

Finally, let us introduce the notion of the cone of critical variations, which we denote by K. The cone K is singled out in $\overline{W}$ by the following conditions:

$$\begin{aligned}
&\dot{\overline{\xi}}(t) = \widehat{\mathbf{r}}'_0(x^0(t)) \overline{\xi}(t) + \overline{z} \widehat{\mathbf{r}}_0(x_0(t)) + \sum_{i=1}^{m-1} \overline{y}_i(t) [\widehat{\mathbf{r}}_0, \widehat{\mathbf{r}}_i](x^0(t)), \\
&\overline{\xi}(t_0^0) = 0, \\
&\overline{\xi}(t_1^0) + \sum_{i=1}^{m-1} \overline{\beta}_i \widehat{\mathbf{r}}_i(x^0(t_1^0)) = 0.
\end{aligned} \tag{7}$$

We can now formulate a condition equivalent to condition (5) in the following form: there is a number $c > 0$ such that

$$\max_{\mathcal{L}eg(\Psi_0)} \omega(\psi(\cdot), \overline{w}) \geqslant c \int_{t_0^0}^{t_1^0} |\overline{y}(t)|^2 \, dt \qquad \forall \overline{w} \in K. \tag{8}$$

Since condition (5) does not depend on the choice of the associated basis, it follows that condition (5) is also independent of this choice. Condition (8) implies that if the form $\omega(\psi(\cdot), \overline{w})$ is positive definite on K for some associated extremal, then condition (8) holds, because in this case we have $\psi(\cdot) \in \mathcal{L}eg(\tau^0)$, and hence the trajectory τ^0 is quadratically rigid. However, below we shall see that this condition is not necessary for the quadratic rigidity.

EXAMPLE

Let $x = (x_1, \dots, x_7)$. Set $e_i = x \mid x_i = 1,\ x_k = 0,\ k \neq i$. By $\widetilde{x}$ we denote the pair (x_4, x_5). Let us define a two-dimensional distribution $\Gamma(x, \varepsilon, n)$ by the following basis:

$$\mathbf{r}_0(x) = \left(\sum_{i=1}^{3} \varphi_i \widetilde{x}\widetilde{x} + \frac{\varepsilon}{4} \theta x_7^2 \right) e_i + \alpha_1(x_6, n) x_7 e_4 + \alpha_2(x_6, n) x_7 e_5 + e_6, \qquad \mathbf{r}_1(x) = e_7. \tag{9}$$

Here φ_i's stand for the forms $\varphi_i \widetilde{x}\widetilde{x} = a_i \frac{1}{2}(x_4^2 - x_5^2) + b_i x_4 x_5 - \varepsilon \frac{1}{2}(x_4^2 + x_5^2)$, $\varepsilon > 0$, $(a_1, b_1) = (1, 0)$, $(a_2, b_2) = (0, 1)$, $(a_3, b_3) = (-1, -1)$, and

$$\alpha_1(x_6, n) = \frac{1}{2} \left(1 + \sin(n x_6) / \sqrt{1/n^2 + \sin^2(n x_6)} \right),$$

$$\alpha_2(x_6, n) = 1 - \alpha_1(x_6, n) = \frac{1}{2} \left(1 - \sin(n x_6) / \sqrt{1/n^2 + \sin^2(n x_6)} \right),$$

where n is a positive integer. Thus, the field $\mathbf{r}_0(x)$ is completely defined. The parameters ε and n will be chosen below. It is clear that $d(\Gamma(x)) = 2$ for all $x \in \mathbb{R}^7$.

Let us define a trajectory τ^0 of a generalized differential equation

$$\dot{x} \in \Gamma(x, \varepsilon, n). \tag{10}$$

We set $\tau^0 = x^0(t)$, where

$$x^0(t) = (x_1^0(t), \dots, x_7^0(t)), \quad x_1^0(t) = x_2^0(t) = x_3^0(t) = x_4^0(t) = x_5^0(t) = 0, \quad x_6^0(t) = t, \quad x_7^0(t) = 0$$

for all t. It is clear that $\dot{x}^0(t) = \mathbf{r}_0(x^0(t)) = e_6$. Therefore, τ^0 is an admissible trajectory of the generalized differential equation (10), and the basis $\mathbf{r}_0(x), \mathbf{r}_1(x)$ is associated to the trajectory τ^0.

Let us consider the trajectory τ^0 on the interval $[0, \pi]$. Let us write out the cone K for $\tau^0 \mid [0, \pi]$. To this end, along with $\mathbf{r}_0(x^0(t))$ and $\mathbf{r}_0'(x^0(t))$, we need the field $[\mathbf{r}_0, \mathbf{r}_1](x^0(t))$. Since $\mathbf{r}_1(x) = e_7$ is a constant field, it follows that $[\mathbf{r}_0, \mathbf{r}_1](x) = \frac{\partial}{\partial x_7} \mathbf{r}_0(x)$. Then (9) implies that

$$[\mathbf{r}_0, \mathbf{r}_1](x) = \sum_{i=1}^{3} \frac{\varepsilon}{2} \theta x_7 e_i + \alpha_1(x_6, n) e_4 + \alpha_2(x_6, n) e_5.$$

Hence, $[\mathbf{r}_0, \mathbf{r}_1](x^0(t)) = \alpha_1(t, n) e_4 + \alpha_2(t, n) e_5$. We can now write out the conditions defining the cone K.

Obviously,

$$\overline{\xi}(t) = \sum_{i=1}^{7} \overline{\xi}(t) e_i, \qquad \mathbf{r}_0(x) = \sum_{i=1}^{7} \mathbf{r}_{0i}(x) e_i.$$

Then, according to (7), we have $\dot{\overline{\xi}}_1 = \mathbf{r}'_{01}(x^0(t))\overline{\xi} + \overline{z}\mathbf{r}_{01}(x^0(t)) = 0$ because $\mathbf{r}_{01}$ is a quadratic form and $x^0(t)$ is the origin of the subspace on which this form is defined. Similarly, $\dot{\overline{\xi}}_2 = \dot{\overline{\xi}}_3 = 0$. This, together with the condition $\overline{\xi}(0) = 0$, yields the relation $\overline{\xi}_1(t) = \overline{\xi}_2(t) = \overline{\xi}_3(t) = 0 \mid [0, \pi]$. Since $\mathbf{r}_{06} = 1$ and $\mathbf{r}_{07} = 0$, it follows that $\dot{\overline{\xi}}_6 = \overline{z}$ and $\dot{\overline{\xi}}_7 = 0$. Hence, $\overline{\xi}_7 = 0 \mid [0, \pi]$. Moreover, $\overline{\xi}(\pi) + \overline{\beta}\mathbf{r}_1 = 0$. Therefore, $\overline{\xi}_6(\pi) = 0$ and $\overline{\beta} = 0$. This yields $\overline{z} = 0$. Thus, $\overline{z} = 0$, $\overline{\beta} = 0$, and $\overline{\xi}_6(t) = \overline{\xi}_7(t) = 0 \mid [0, \pi]$. Finally, $\dot{\overline{\xi}}_4 = \mathbf{r}'_{04}(x^0(t))\overline{\xi}(t) + \alpha_1(t, n)\overline{y}(t)$. However,

$$\mathbf{r}'_{04}(x^0(t))\overline{\xi}(t) = \frac{\partial}{\partial x_7}\mathbf{r}_{04}(x^0(t))\,\overline{\xi}_7(t) = 0.$$

Thus, $\dot{\overline{\xi}}_4 = \alpha_1(t, n)\overline{y}(t)$. It follows from the boundary conditions that $\overline{\xi}_4(0) = \overline{\xi}_4(\pi) = 0$.

We can similarly obtain $\dot{\overline{\xi}}_5 = \alpha_2(t, n)\overline{y}(t)$ and $\overline{\xi}_5(0) = \overline{\xi}_5(\pi) = 0$.

Thus the cone K is completely described. Summing up, we obtain

$$\begin{gathered} K: \quad \overline{\beta} = 0;\ \overline{z} = 0;\ \overline{\xi}_i(t) = 0 \mid [0, \pi], \quad i \neq 4, 5, \\ \dot{\overline{\xi}}_4 = \alpha_1(t, n)\overline{y}(t), \quad \overline{\xi}_4(0) = \overline{\xi}_4(\pi) = 0, \qquad \dot{\overline{\xi}}_5 = \alpha_2(t, n)\overline{y}(t), \quad \overline{\xi}_5(0) = \overline{\xi}_5(\pi) = 0. \end{gathered} \tag{11}$$

We must now write out the set $\Psi_0(\tau^0)$. We certainly have $\psi(t) = (\psi_1(t), \dots, \psi_7(t))$. By setting

$$H(\psi(t), x(t), u^0(t)) = \psi\mathbf{r}_0(x) = \sum_{i=1}^{7} \psi_i \mathbf{r}_{0i}(x),$$

we obtain $H'_{x_i}(\psi(t), x^0(t), u^0(t)) = 0$ (for $i \neq 7$) for any ψ and t. Hence, $\psi_1 = \text{const}$, $\psi_2 = \text{const}$, $\psi_3 = \text{const}$, $\psi_4 = \text{const}$, $\psi_5 = \text{const}$, and $\psi_6 = \text{const}$. This yields

$$-\dot{\psi}_7 = H'_{x_7}(\psi, x^0(t)) = \psi_4\alpha_1(t, n) + \psi_5\alpha_2(t, n).$$

However, $\psi(t)\mathbf{r}_1 = \psi_7$. Hence, $\psi_7 = 0$. Therefore, $\psi_4\alpha_1(t, n) + \psi_5\alpha_2(t, n) = 0 \mid [0, \pi]$. Consequently, $\psi_4 = \psi_5 = 0$. Finally, $\psi(t)\mathbf{r}_0(x^0(t)) = \psi_6$, and thus $\psi_6 = 0$. For the normalizing condition we take $\psi_1^2 + \psi_2^2 + \psi_3^2 = 1$.

We see that $\Psi_0(\tau^0)$ is of the form

$$\psi_1, \psi_2, \psi_3 = \text{const}; \qquad \psi_1^2 + \psi_2^2 + \psi_3^2 = 1 \qquad (\psi_i = 0,\ i > 3). \tag{12}$$

All the conditions are certainly satisfied. In what follows, for brevity we set $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $\lambda_i = \psi_i$, $i = 1, 2, 3$, and identify ψ with λ.

Let us find the set $\mathcal{L}eg(\Psi_0(\tau^0))$. We have

$$[[\mathbf{r}_0, \mathbf{r}_1], \mathbf{r}_1](x) = \frac{\partial^2 \mathbf{r}_0(x)}{\partial x_7^2} = \frac{\varepsilon}{2}\theta \sum_{i=1}^{3} e_i.$$

Hence,

$$\psi \in \mathcal{L}eg(\Psi_0(\tau^0)) \Longleftrightarrow \sum_{i=1}^{3} \lambda_i \geqslant 0. \tag{13}$$

It remains to write out the form $\omega(\lambda,\overline{w})$ on the cone K. Since $\overline{\beta}\mid K=0$, it follows that $\omega_1(\lambda;\overline{w})=0\mid K$. Let us pass to ω_2. Denote by $\tilde{\overline{\xi}}(t)$ the collection $\tilde{\overline{\xi}}(t)=(\overline{\xi}_4(t),\overline{\xi}_5(t))$. Further,

$$\psi\mathbf{r}_0(x)=\sum_{i=1}^{3}\lambda_i\mathbf{r}_{0i}(x).$$

Hence, on K we have

$$\frac{1}{2}(\psi\mathbf{r}_0)''(x^0(t))\overline{\xi}(t)\overline{\xi}(t)=\sum_{i=1}^{3}\lambda_i\varphi_i\tilde{\overline{\xi}}(t)\tilde{\overline{\xi}}(t).$$

We can readily see that the other terms are zero, except for the last one. The last term is equal to $(\varepsilon/4)\theta\overline{y}^2(t)\sum_{i=1}^{3}\lambda_i$. Thus,

$$\omega_2(\lambda;\overline{w})(t)=\sum_{i=1}^{3}\lambda_i\varphi_i\tilde{\overline{\xi}}(t)\tilde{\overline{\xi}}(t)+\frac{\varepsilon}{4}\theta\sum_{i=1}^{3}\lambda_i\overline{y}^2(t).$$

Hence, on K we finally have

$$\omega(\lambda;\overline{w})=\int_0^{\pi}\left(\sum_{i=1}^{3}\lambda_i\varphi_i\tilde{\overline{\xi}}(t)\tilde{\overline{\xi}}(t)+\frac{\varepsilon}{4}\theta\overline{y}^2(t)\sum_{i=1}^{3}\lambda_i\right)dt. \tag{14}$$

The sufficient condition (8) is formulated for $\tilde{\overline{\xi}}(t),\overline{y}(t)$, where $\tilde{\overline{\xi}}(t),\overline{y}(t)$ are connected by formulas (11). This condition has the following form: there is a number $c>0$ such that

$$\max_{\lambda\mid\Sigma\lambda_i^2=1,\,\Sigma\lambda_i\geqslant 0}\omega(\lambda;\overline{w})\geqslant c\int_0^{\pi}\overline{y}^2(t)\,dt\quad\forall\overline{w}\in K. \tag{15}$$

We can now pass to the main results.

Proposition 1. *For all sufficiently small $\varepsilon>0$, the trajectory $\tau^0\mid[0,\pi]$ is quadratically rigid.*

Proof. Set $z=(z_1,z_2)\in\mathbb{R}^2$. Denote by $\widehat{Z}$ the set of absolutely continuous functions on $[0,\pi]$ such that $\int_0^{\pi}\dot{z}^2\,dt<+\infty$ and $z(0)=z(\pi)=0$. We denote elements of the set $\widehat{Z}$ by $\widehat{z}$. Set

$$\widehat{\sigma}(\widehat{z})=\frac{1}{2}\int_0^{\pi}(z_1^2+z_2^2)\,dt,\qquad\widehat{\kappa}(\widehat{z})=\frac{1}{2}\int_0^{\pi}(z_1^2(t)-z_2^2(t))\,dt,\qquad\widehat{\eta}(\widehat{z})=\int_0^{\pi}z_1(t)z_2(t))\,dt.$$

It is clear that there is a relationship connecting $\widehat{\sigma}(\widehat{z})$, $\widehat{\kappa}(\widehat{z})$, and $\widehat{\eta}(\widehat{z})$, namely, $\widehat{\kappa}^2+\widehat{\eta}^2\leqslant\widehat{\sigma}^2$, which is equivalent to the Schwartz inequality. In all other relations, the possible values of $\widehat{\sigma}(\widehat{z})$, $\widehat{\kappa}(\widehat{z})$, and $\widehat{\eta}(\widehat{z})$ are arbitrary. Let numbers σ, κ, and η be given such that $\kappa^2+\eta^2\leqslant\sigma^2$. Set $J(\sigma,\,\kappa,\,\eta)=\min\int_0^{\pi}\dot{z}^2\,dt$, where the minimum is taken over all $\widehat{z}=z(t)\mid[0,\pi]$ such that $\widehat{\sigma}(\widehat{z})=\sigma$, $\widehat{\kappa}(\widehat{z})=\kappa$, and $\widehat{\eta}(\widehat{z})=\eta$. A solution of this problem exists and can readily be found by the maximum principle.

It turns out that

$$J(\sigma,\,\kappa,\,\eta)=5\sigma-3\sqrt{\kappa^2+\eta^2}. \tag{16}$$

We do not discuss this problem here.

Set

$$\Phi_i(\widehat{z}) = \int_0^\pi \Big(\varphi_i z(t) z(t) + \frac{\varepsilon}{4}\theta \dot{z}^2(t)\Big)\, dt.$$

It follows from the definition of φ_i that

$$\Phi_i(\widehat{z}) = a_i \widehat{\kappa}(\widehat{z}) + b_i \widehat{\eta}(\widehat{z}) - \varepsilon\widehat{\sigma}(\widehat{z}) + \frac{\varepsilon}{4}\theta \int_0^\pi \dot{z}^2(t)\, dt.$$

Set $\Phi(\widehat{z}) = \max\limits_{i=1,2,3} \Phi_i(\widehat{z})$. Hence,

$$\Phi(\widehat{z}) = \max_i (a_i \widehat{\kappa}(\widehat{z}) + b_i \widehat{\eta}(\widehat{z})) - \varepsilon\widehat{\sigma}(\widehat{z}) + \frac{\varepsilon}{4}\theta \int_0^\pi \dot{z}^2(t)\, dt.$$

It follows from the choice of (a_i, b_i), $i = 1, 2, 3$, that the inequality

$$\max_i (a_i \overline{\kappa} + b_i \overline{\eta}) \geqslant \frac{1}{2\sqrt{2}} \sqrt{\overline{\kappa}^2 + \overline{\eta}^2}$$

holds for any $\overline{\kappa}$ and $\overline{\eta}$. Therefore,

$$\Phi(\widehat{z}) \geqslant \frac{1}{2\sqrt{2}} \sqrt{\widehat{\kappa}^2(\widehat{z}) + \widehat{\eta}^2(\widehat{z})} - \varepsilon\widehat{\sigma}(\widehat{z}) + \frac{\varepsilon}{4}\theta \int_0^\pi \dot{z}^2(t)\, dt.$$

Let $\theta' < 1$ and $\theta\theta' > 4/5$. It is clear that

$$\Phi(\widehat{z}) \geqslant \frac{1}{2\sqrt{2}} \sqrt{\widehat{\kappa}^2(\widehat{z}) + \widehat{\eta}^2(\widehat{z})} - \varepsilon\widehat{\sigma}(\widehat{z}) + \frac{\varepsilon}{4}\theta\theta' \int_0^\pi \dot{z}^2(t)\, dt + \frac{\varepsilon}{4}\theta(1-\theta') \int_0^\pi \dot{z}^2(t)\, dt.$$

Applying relation (16), we obtain

$$\Phi(\widehat{z}) \geqslant \frac{1}{2\sqrt{2}} \sqrt{\widehat{\kappa}^2(\widehat{z}) + \widehat{\eta}^2(\widehat{z})} - \varepsilon\widehat{\sigma}(\widehat{z}) + \frac{\varepsilon}{4}\theta\theta' \Big(5\widehat{\sigma}(\widehat{z}) - 3\sqrt{\widehat{\kappa}^2(\widehat{z}) + \widehat{\eta}^2(\widehat{z})}\Big) + \frac{\varepsilon}{4}\theta(1-\theta') \int_0^\pi \dot{z}^2(t)\, dt$$

$$= \left(\frac{1}{2\sqrt{2}} - 3\frac{\varepsilon}{4}\theta\theta'\right) \sqrt{\widehat{\kappa}^2(\widehat{z}) + \widehat{\eta}^2(\widehat{z})} + \varepsilon\Big(\frac{5}{4}\theta\theta' - 1\Big)\widehat{\sigma}(\widehat{z}) + \frac{\varepsilon}{4}\theta(1-\theta') \int_0^\pi \dot{z}^2(t)\, dt.$$

Hence, for $\varepsilon < 2/(3\sqrt{2}\theta\theta')$ we have

$$\Phi(\widehat{z}) \geqslant \frac{\varepsilon}{4}\theta(1-\theta') \int_0^\pi \dot{z}^2(t)\, dt. \tag{17}$$

Let $\varepsilon > 0$ be such that relation (17) holds. Let $\overline{w} \in K$; then $\overline{\widetilde{\xi}}(t) \in \widehat{Z}$. Set $\overline{\widetilde{\xi}}(t) = \widehat{z}(\overline{w})$. Substituting $\widehat{z}(\overline{w})$ into $\Phi(\widehat{z})$ and applying (17), we see that

$$\Phi(\widehat{z}(\overline{w})) \geqslant \frac{\varepsilon}{4}\theta(1-\theta') \int_0^\pi \Big(\dot{\overline{\widetilde{\xi}}}\Big)^2 (t)\, dt.$$

Replacing $\dot{\overline{\xi}}(t)$ according to (11), we obtain

$$\Phi(\widehat{z}(\overline{w})) \geqslant \frac{\varepsilon}{4}\theta(1-\theta')\int\limits_0^\pi (\alpha_1^2(t,n)+\alpha_2^2(t,n))\overline{y}^2\,dt.$$

However, $\alpha_1(t,n)$, $\alpha_2(t,n) > 0$, and $\alpha_1(t,n)+\alpha_2(t,n)=1$. Hence,

$$\Phi(\widehat{z}(\overline{w})) \geqslant \frac{\varepsilon}{8}\theta(1-\theta')\int\limits_0^\pi \overline{y}^2\,dt.$$

Set $\lambda^i = \{\lambda \mid \lambda_i = 1,\ \lambda_k = 0,\ k \neq i\}$. It is clear that $\lambda^i \in \mathcal{L}eg(\Psi_0(\tau^0))$, $i=1,2,3$. Then

$$\omega(\lambda^i,\overline{w}) = a_i\widehat{\kappa}(\widehat{z}(\overline{w})) + b_i\widehat{\eta}(\widehat{z}(\overline{w})) - \varepsilon\widehat{\sigma}(\widehat{z}(\overline{w})) + \frac{\varepsilon}{4}\theta\int\limits_0^\pi \overline{y}^2(t)\,dt.$$

This yields $\omega(\lambda^i,\overline{w}) \geqslant \Phi_i(\widehat{z}(\overline{w}))$. Hence,

$$\max_i\ \omega(\lambda^i,\overline{w}) \geqslant \Phi(\widehat{z}(\overline{w})) \geqslant \frac{\varepsilon}{8}\theta(1-\theta')\int\limits_0^\pi \overline{y}^2\,dt.$$

This proves that, for any sufficiently small $\varepsilon > 0$, the trajectory $\tau^0 \mid [0,\pi]$ satisfies condition (8). This completes the proof of Proposition 1.

Let $\varepsilon > 0$ be such that relation (17) holds. In this case, as was proved above, the trajectory $\tau^0 \mid [0,\pi]$ is quadratically rigid.

Proposition 2. *For all sufficiently large n, the form $\omega(\lambda,\overline{w})$ is alternating on K for any value $\lambda \in \Psi_0(\tau^0)$.*

Proof. It suffices to show that, starting from some $n = n_*$, the form $\omega(\lambda,\overline{w})$ takes negative values on K for any $\lambda \in \Psi_0(\tau^0)$.

If $\sum_{i=1}^3 \lambda_i < 0$, then the desired assertion holds for any n. Indeed, let n be chosen and let $\{\overline{y}^s(t)\}_{s=1,2,\dots}$ be a sequence of functions defined on $[0,\pi]$ and such that $\int_0^\pi \overline{y}^{s\,2}\,dt = 1$ and $\overset{\approx}{\xi}{}^s(0) = \overset{\approx}{\xi}{}^s(\pi) = 0$, where $\overset{\approx}{\xi}{}^s(t)$ and $\overline{y}^s(t)$ are connected by relation (11), and the sequence $\{\overline{y}^s(t)\}$ is weakly convergent to zero. Then $\overset{\approx}{\xi}{}^s(t)$ is uniformly convergent to zero on $[0,\pi]$, and hence

$$\omega(\lambda,\overline{w}^s) \underset{s\to\infty}{\longrightarrow} \frac{\varepsilon\theta}{4}\sum_{i=1}^3 \lambda_i, \tag{18}$$

where $\overline{w}^s$ is an element of K defined by the elements $\overset{\approx}{\xi}{}^s(t)$ and $\overline{y}^s(t)$. It follows from (18) that $\omega(\lambda,\overline{w})$ takes negative values on K. Thus, the case $\sum_{i=1}^3 \lambda_i < 0$ is completely studied.

Let us consider the case $\sum_{i=1}^3 \lambda_i > 0$. Introduce some notation. Denote a pair (κ,η) by μ and a pair (a_i,b_i) by ρ^i, $i=1,2,3$. Then $a_i\kappa + b_i\eta = \rho^i\mu$. Finally, we set $\rho_\lambda = \sum_{i=1}^3 \lambda_i\rho^i$. Let $\overline{w} \in K$; then

$$\omega(\lambda,\overline{w}) = \rho_\lambda\widehat{\mu}(\widehat{z}(\overline{w})) - \varepsilon\widehat{\sigma}(\widehat{z}(\overline{w}))\sum_{i=1}^3 \lambda_i + \frac{\varepsilon}{4}\theta\int\limits_0^\pi \overline{y}^2(t)\,dt. \tag{19}$$

Let $|\mu| = \sqrt{\kappa^2 + \eta^2} = 1$ and

$$\overline{y}_n(t,\mu) = \alpha_1(t,n)\left(\sqrt{1+\kappa}\sqrt{\frac{2}{\pi}}\cos t + \delta_{1n}(\mu)\right) + \alpha_2(t,n)\left(\operatorname{sign}\eta\sqrt{1-\kappa}\sqrt{\frac{2}{\pi}}\cos t + \delta_{2n}(\mu)\right).$$

The constants $\delta_{1n}(\mu)$ $\delta_{2n}(\mu)$ can be found from the relations

$$\int_0^\pi \alpha_1(t,n)\overline{y}_n(t,\mu)\,dt = \int_0^\pi \alpha_2(t,n)\overline{y}_n(t,\mu)\,dt = 0.$$

It follows from the same relations that $\overline{y}_n(t,\mu)$ defines an element $\overline{w}_n(\mu) \in K$.

We can see that $\alpha_1^2(t,n)$ and $\alpha_2^2(t,n)$ are weakly convergent to $1/2$ while $\alpha_1(t,n)\alpha_2(t,n)$ is weakly convergent to zero. Hence, for $\widehat{z}(\overline{w}_n(\mu))$ we have the representation

$$\begin{gathered}\widehat{z}(\overline{w}_n(\mu)) = (z_{1n}(t,\mu),\, z_{2n}(t,\mu)),\\ z_{1n}(t,\mu) = \frac{1}{2}\sqrt{1+\kappa}\sqrt{\frac{2}{\pi}}\sin t + \delta_{1n}(t,\mu), \qquad z_{2n}(t,\mu) = \frac{1}{2}\operatorname{sign}\eta\sqrt{1-\kappa}\,\sin t + \delta_{2n}(t,\mu),\end{gathered} \tag{20}$$

where

$$\max_{t,\mu | t\in[0,\pi], |\mu|=1} |\delta_{1n}(t,\mu)| \to 0 \quad \text{as} \quad n\to\infty, \qquad \max_{t,\mu | t\in[0,\pi], |\mu|=1} |\delta_{2n}(t,\mu)| \to 0 \quad \text{as} \quad n\to\infty.$$

Moreover,

$$\int_0^\pi \overline{y}_n^2(t,\mu)\,dt = 1 + \varepsilon_n(\mu), \tag{21}$$

where $\max_{\mu|\,|\mu|=1} |\varepsilon_n(\mu)| \to 0$ as $n\to\infty$. Applying relation (20), we obtain

$$\widehat{\mu}(\widehat{z}(\overline{w}_n(\mu))) = \mu/4 + \overline{\varepsilon}_{1n}(\mu), \qquad \widehat{\sigma}(\widehat{z}(\overline{w}_n(\mu))) = 1/4 + \overline{\varepsilon}_{2n}(\mu), \tag{22}$$

where

$$\max_{\mu||\mu|=1} |\overline{\varepsilon}_{1n}(\mu)|; |\overline{\varepsilon}_{2n}(\mu)| \to 0 \quad \text{as} \quad n\to\infty.$$

Thus, substituting $\overline{w}_n(\mu)$ into (19) and using (21) and (22), we obtain

$$\omega(\lambda,\overline{w}_n(\mu)) = \frac{1}{4}\rho_\lambda\mu + \sum_{i=1}^3 \lambda_i\left(-\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\theta\right) + \overline{\varepsilon}_{3n}(\mu,\lambda), \tag{23}$$

where

$$\max_{\mu||\mu|=1,\,\lambda\in\Psi_0,\,\Sigma\lambda\geqslant 0} |\overline{\varepsilon}_{3n}(\mu,\lambda)| \to 0 \quad \text{as} \quad n\to\infty.$$

We set $\mu(\lambda) = \mu \mid \rho_\lambda\mu = \min_{|\mu'|=1} \rho_\lambda\mu'$. Then $\rho_\lambda\mu(\lambda) = -|\rho_\lambda|$. Substituting $\overline{w}_n(\mu(\lambda))$ into (23), we obtain

$$\omega(\lambda,\overline{w}_n(\mu(\lambda))) = -\frac{1}{4}|\rho_\lambda| - \frac{\varepsilon}{4}(1-\theta)\sum_{i=1}^3 \lambda_i + \overline{\varepsilon}_{3n}(\mu(\lambda),\lambda). \tag{24}$$

However, $-\frac{1}{4}|\rho_\lambda| - \frac{\varepsilon}{4}(1-\theta)\sum_{i=1}^3 \lambda_i \leqslant -c < 0$ for any λ such that $\lambda\in\Psi_0$ and $\sum\lambda \geqslant 0$.

Let n_* be such that
$$\max_{\lambda} |\overline{\varepsilon}_{3n}(\mu(\lambda),\lambda)| < |c| \quad \forall n \geqslant n_*.$$
Then it follows from (24) that, for $n \geqslant n_*$, we obtain $\omega(\lambda, \overline{w}_n(\mu(\lambda))) < 0$ for any λ such that $\sum_{i=1}^{3} \lambda_i^2$ and $\overline{\lambda}$ are nonnegative. We have thus proved that, for $n \geqslant n_*$, the form ω takes negative values on K for any $\lambda \in \Psi_0(\tau^0)$. This readily implies the desired assertion and completes the proof of Proposition 2.

It follows from Propositions 1 and 2 that we can choose the parameters ε and n in (9) so that the trajectory $\tau^0|[0,\pi]$ becomes rigid with respect to the distribution $\Gamma(x,\varepsilon,n)$ but, for any associated extremal, the form ω is alternating on the cone K of critical variations, i.e., has the conjugate point inside the interval $[0,\pi]$.

Thus, in condition (8), the set $\mathcal{L}eg(\tau^0)$ cannot be replaced by a single element without making the sufficient condition rougher.

It remains to show that the distribution $\Gamma(x,\varepsilon,n)$ is bracket-generated for any $\varepsilon > 0$ and $n \geqslant 1$. We only sketch the proof.

According to (9), we have
$$[\mathbf{r}_0,\mathbf{r}_1](x) = \frac{\partial}{\partial x_7}\mathbf{r}_0(x) = \sum_{i=1}^{3} \frac{\varepsilon\theta}{2} x_7 e_i + \alpha_1(x_6,n)e_4 + \alpha_2(x_6,n)e_5, \qquad [[\mathbf{r}_0,\mathbf{r}_1],\mathbf{r}_1](x) = \frac{\varepsilon\theta}{2}\sum_{i=1}^{3} e_i.$$
Combining the fields $\mathbf{r}_0$ and $[\mathbf{r}_0,\mathbf{r}_1]$ with the field $\sum_{i=1}^{3} e_i$, we obtain the fields
$$\rho_1 = \sum_{i=1}^{3} e_i, \quad \rho_2 = \alpha_1(x_6,n)e_4 + \alpha_2(x_6,n)e_5, \quad \rho_3 = \sum_{i=1}^{3} \varphi_i \widetilde{x}\widetilde{x} e_i + e_6, \quad \rho_4 = e_7.$$
Moreover,
$$[\rho_2,\rho_3](x) = \frac{\partial}{\partial x_6}\rho_2 - \sum_{i=1}^{3} 2\varphi_i \widetilde{x}\alpha(x_6,n)e_i,$$
where $\alpha(x_6,n) = (\alpha_1(x_6,n),\, \alpha_2(x_6,n))$, and $\rho_5 = [\rho_2,[\rho_2,\rho_3]](x) = \sum_{i=1}^{3} 2\varphi_i\alpha(x_6,n)\alpha(x_6,n)e_i$. This yields
$$\rho_{5,k} = \underbrace{[\cdots[}_{k}\rho_5,\rho_3],\rho_3\underbrace{]\cdots}_{k},\rho_3] = \sum_{i=1}^{3} \frac{\partial^k}{\partial x_6^k} 2\varphi_i\alpha(x_6,n)\alpha(x_6,n)e_i, \quad k = 1,2,\dots.$$
However, the linear span of the fields $\sum_{i=1}^{3} e_i$ and $\sum_{i=1}^{3} 2\varphi_i\widetilde{x}\widetilde{x}e_i$ is $\mathbb{R}^3 = \mathrm{Lin}(e_1,e_2,e_3)$. Since φ_i analytically depends on α, and α analytically depends on x_6, it follows that, at each point x_6, the linear span of the sequence $\sum\limits_{i=1}^{3} e_i, \rho_5, \rho_{5,1}, \dots, \rho_{5,k}, \dots$ coincides with $\mathbb{R}^3$. Hence, the fields e_1, e_2, and e_3 are bracket-generated. Then, combining these fields with the field ρ_3, we see that the field e_6 is bracket-generated as well. Therefore,
$$\rho_{2,k} = \underbrace{[\cdots[}_{k}\rho_2,e_6],e_6\underbrace{]\cdots}_{k},e_6] = \frac{\partial^k \rho_2}{\partial x_6^k} \qquad (k = 1,2,\dots).$$
However, the span of the field $\rho_2(x)$ over all x_6 coincides with the linear span of e_4 and e_5. Then, since ρ_2 is analytic with respect to x_6, it follows that, at each point x_6, the linear span of the fields ρ_2 and $\rho_{2,k}$ $(k = 1,2\dots)$ coincides with $\mathrm{Lin}(e_4,e_5)$. Therefore, at each point, the fields e_4 and e_5 are bracket-generated.

Thus, we proved that each of the fields e_i $(i = 1,\dots,7)$ is bracket-generated. For this reason, $\Gamma(x,\varepsilon,n)$ is also bracket-generated.

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