A.V. DMITRUK and A.A. MILYUTIN

A Condition of Legendre Type for Optimal Control Problems, Linear in the Control

Abstract

For singular extremals in the above class of problems conditions of a Pontryagin minimum include a new pointwise condition, i.e. a condition of Legendre type. It involves not only the second, but also the third variation of Lagrange function, and requires to solve a nontrivial auxiliary optimal control problem. Some cases of its exact solution are presented.

Key words: singular extremal, weak and Pontryagin minimum, quadratic order of estimation, necessary and sufficient conditions, second and third variations of Lagrange function.

1 Introduction

For the control system, linear in the control, and nonlinear in the state variable,

$$\dot{x} = f(x,t) + F(x,t)\,u, \tag{1}$$

on a fixed time interval $[0,T]$, where both the control and state variables are multidimensional, we consider the following optimal control problem: to minimize a terminal functional $J = \varphi_0(p)$ subject to terminal constraints $\varphi_i(p) \le 0,\ i = 1,\dots\nu,\ \ g(p) = 0,$ and to a pointwise control constraint $u(t) \in U(t)$.

Here $p = (x(0), x(T)),\ \ \dim\, g = q,$ and $U(t)$ is a convex solid set, Hausdorff continuous in t. All the data functions are assumed to be at least C^2- smooth in $x,\, p,$ and C^1- smooth in t. This is a fairly general statement, which includes (after simple reformulations) problems with integral functionals and integral constraints, time-optimal problems, and others.

Denote by $W = AC^m \times L^r_\infty[0,T]$ the space of all pairs of functions $w = (x,u)$, where $x(t)$ is absolutely continuous $m-$ dimensional, and $u(t)$ is measurable essentially bounded $r-$ dimensional, and let $w^0 = (x^0, u^0)$ be an examined extremal.

We assume here that $u^0(t)$ goes strictly inside $U(t)$. Then obviously w^0 is a singular extremal, and hence it would not be too restrictive to assume also that $u^0(t)$ is continuous. (In a generic case, differentiating two times the switching function, one can express $u^0(t)$ in terms of the state and costate variables.)

For such extremals the classical Legendre–Clebsch condition holds trivially: the corresponding term obviously vanishes. However, there are some other informative pointwise conditions, which, like any pointwise conditions, we call conditions of Legendre type. They are part of a full set of necessary conditions as well as of sufficient ones for a local minimum. In papers [10–15] close pairs of second-order necessary and sufficient conditions were obtained for minimum of the following two types: a weak and so-called Pontryagin minimum (briefly, $\Pi-$ minimum). We recall them in the next section.

We say that w^0 is *a Pontryagin minimum point* in the above problem, if it is a point of L_1 -minimum with respect to the control on any uniformly bounded control set. Obviously, the $\Pi-$ minimum occupies an intermediate position between the classic weak and strong minima. Note that for a weak minimum one may ignore the control constraint $u(t) \in U(t)$, but for a $\Pi-$ minimum one may not do that.

2 Quadratic order optimality conditions

Let $\Lambda = \{\lambda = (\alpha, \beta, \psi)\}$, where $\alpha = (\alpha_0, \ldots, \alpha_\nu) \geq 0$, $\beta \in R^q$, and $\psi(t)$ is a Lipschitzian $m-$ dimensional function (costate variable), be the set of all normalized collections of Lagrange multipliers, providing the fulfillment of the Pontryagin Maximum Principle for the given extremal w^0. For any $\lambda \in \Lambda$ denote by $\Omega[\lambda](\bar{w})$ the second variation of the corresponding Lagrange function

$$\Phi(\bar{w}) = \sum_0^\nu \alpha_i \varphi_i(p) + \beta g(p) + \int (\psi, \dot{x} - f(x,t) + F(x,t)\, u)\, dt \quad \text{at} \ w^0 .$$

The role of Legendre conditions is to select these λ from Λ to a subset $M(\Lambda)$, generating the functional

$$\Omega[M(\Lambda)](\bar{w}) = \sup \{\Omega[\lambda](\bar{w}) \mid \lambda \in M(\Lambda)\},$$

by means of which quadratic order conditions of local minimum are formulated.

NECESSARY CONDITION:

$$\Omega[M(\Lambda)](\bar{w}) \ \geq \ 0 \qquad \text{on the critical cone } \mathcal{K}; \tag{2}$$

SUFFICIENT CONDITION: $\exists\, a > 0$, such that

$$\Omega[M(\Lambda)](\bar{w}) \ \geq \ a\gamma(\bar{w}) \qquad \text{on the critical cone } \mathcal{K}, \tag{3}$$

where $\mathcal{K}$ is defined by linearization of all the constraints at w^0, and γ is the following quadratic functional ("order") of estimation [11–15]:

$$\gamma(\bar{w}) = |\bar{x}(t_0)|^2 + |\bar{y}(t_1)|^2 + \int_0^T |\bar{y}(t)|^2 \, dt. \tag{4}$$

$$\dot{\bar{y}} = \bar{u}, \qquad \bar{y}(0) = 0, \tag{5}$$

It is worth to note, that γ contains (in addition to endpoint terms) only the integral of the squares of state variations ($\bar{x}$ and $\bar{y}$), but not variations of the control itself.

Observe, that these necessary and sufficient conditions are maximally close to each other with respect to the order γ. We say that they constitute an *adjoining pair of conditions.* In this sense conditions (2) and (3) are similar to the conditions from the finite-dimensional calculus and the classical calculus of variations.

Remarkably, the Legendre conditions and hence the subset $M(\Lambda)$ depend on the examined type of minimum at w^0. For the $\Pi-$ minimum as compared with the weak minimum there is an additional Legendre condition, i.e. an additional pointwise test for λ while selecting $M(\Lambda)$ (condition (12) below), and hence the subset $M(\Lambda)$ is smaller. In a typical case when Lagrange multipliers are unique, i.e. Λ consists of a single λ , we get that Lagrange function, corresponding to this λ , must satisfy the additional pointwise condition (12).

Note that for a weak minimum Legendre type conditions concern only the second variation of Lagrange function (a lot of them were obtained by many authors, see e.g. [1–10]; as the most important among them we regard Goh conditions of equality and inequality type [2]), and by means of the so-called Goh transformation these conditions come in principle to the classical Legendre condition with respect to a new control variable (while the initial problem can not generally be reduced in full to the new control).

Recall this transformation for quadratic forms relevant to our problem. For any $\lambda,$ the second variation is a quadratic form of the type:

$$\Omega(\bar{w}) = (S\bar{x}_T, \bar{x}_T) + \int_0^T [(D(t)\bar{x}, \bar{x}) + (P(t)\bar{x}, \bar{u})] \, dt,$$

where

$$\dot{\bar{x}} = A(t)\bar{x} + B(t)\bar{u}, \tag{6}$$

$\bar{w} = (\bar{x}, \bar{u}) \in W,$ the matrices A, Q, R are measurable and essentially bounded. Assuming that $P,\ B$ are Lipschitz continuous, the Goh transformation is: $(\bar{x}, \bar{u}) \to (\bar{\xi}, \bar{y}, \bar{u}),$ where $\bar{\xi} = \bar{x} - B\bar{y},$ and $\bar{y}$ satisfies (5), so that

$$\dot{\bar{\xi}} = A\bar{\xi} + B_1\bar{y}, \qquad \bar{\xi}(0) = 0, \tag{7}$$

and Ω takes the form:

$$\Omega(\bar\xi,\bar y,\bar u) = \ (S_1(\bar\xi_T,\bar y_T),(\bar\xi_T,\bar y_T)) +$$

$$+\int_0^T [(D_1(t)\bar\xi,\bar\xi) + (P_1(t)\bar\xi,\bar y) + (Q(t)\bar y,\bar y) + (V(t)\bar y,\bar u)]\,dt, \tag{8}$$

where $B_1 = AB - \dot B,$ and the matrix V is Lipschitzian skew-symmetric. The term $(C(t)\bar\xi,\bar u)$ is reduced to the presented ones using the integration by parts in view of (5) and (7). If Ω is nonnegative on the subspace (6), or even on some its subspace of a finite codimension, then the following Goh conditions hold:

$$a)\ V(t)=0, \qquad b)\ Q(t)\ge 0. \tag{9}$$

Note that if the first of these conditions holds, then the meaning of the last one is very simple: in fact, it is the classical Legendre condition w.r.t. $\bar y$ for quadratic form (8), in which one can consider $\bar y$ as a new control variable (see details in [12–14]). If we denote by $G(\Lambda)$ the set of all $\lambda\in\Lambda,$ such that the corresponding second variation $\Omega[\lambda](\bar w)$ satisfies conditions (9), then in conditions (2, 3) for the weak minimum one can take as $M(\Lambda)$ the set $G(\Lambda)$.

If it happens that $Q(t)=0$ on some interval, and, again, P_1, B_1 are Lipschitz continuous, then Goh transformation can be repeated, and so on. Iterations of this transformation were generalized by A.A.Milyutin in an abstract setting [10], which yields the following result.

Denote by Λ^+ the set of all $\lambda\in\Lambda,$ such that the corresponding second variation $\Omega[\lambda](\bar w)$ is nonnegative on a subspace in W (depending on λ) of a finite codimension. Obviously, $G(\Lambda)\subset\Lambda^+$.

Theorem 1 (A.A.Milyutin). *Let w^0 be a weak minimum point in the problem. Then Λ^+ is nonempty, and*

$$\Omega[\Lambda^+](\bar w)\ge 0 \quad \textit{for all} \quad \bar w\in\mathcal K.$$

One can easily show that if $\lambda\in\Lambda^+$, then $\Omega[\lambda]$ satisfies Goh conditions for all possible iterations of Goh transformation. Due to this fact, Theorem 1 strengthens, in particular, results of [1–9]. We do not expose this in detail, since this is not the aim of our paper.

Now let us pass to the Pontryagin minimum. Here the situation is essentially different. In this case a new Legendre type condition arises, which has rather unusual formulation and, being pointwise, however leads to an auxiliary optimal control problem, involving the second and the third (!) variations of Lagrange function, and the admissible control set U (results of [11–15]).

3 The new Legendre type condition

For any $\lambda = (\alpha, \beta, \psi)$ consider the corresponding Pontryagin function $H[\lambda](x,u,t) = \psi(t)[f(x,t) + F(x,t)u]$, and introduce the following cubic functional:

$$\rho[\lambda](\bar w) \;=\; \int_0^T [-(H_{uxx}[\lambda]\,\bar x, \bar x, \bar u) + 2((F'_x\,\bar x, \bar u), H_{xu}[\lambda]\,\bar y)\,]\,dt. \qquad (10)$$

It is the third variation of Lagrange function at w^0 on equation (1) to within $o(\gamma)$ on Pontryagin variations (i.e. sequences $(\bar x_n, \bar u_n)$, such that $\|\,\bar x_n\,\|_C + \|\,\bar u_n\,\|_1 \to 0,$ and $\|\,\bar u_n\,\|_\infty \le$ const), see [12–14] for details. Without loss of generality, below we consider $w^0 = 0$ and omit the bar over the variables $x, u, ...$ Plugging here $x = \xi + By,$ with $B(t) = F(x^0(t), t)$ (Goh transformation), and collecting similar terms, we get

$$\rho[\lambda](w) = \int [(T_1[\lambda](t)\xi, \xi, u) + (T_2[\lambda](t)\xi, y, u) + (\mathcal{E}[\lambda](t)y, y, u)]\,dt,$$

where T_1, T_2, $\mathcal{E}$ are cubic tensors of corresponding dimensions. Here the two first terms are $o(\gamma)$ on Pontryagin variations, hence only the last term is actually essential. (Formally, this term can be obtained from (10) simply by putting $\bar x = B\bar y$.) Thus, we come to the cubic functional

$$e[\lambda](w) = \int (\mathcal{E}[\lambda](t)\,y, y, u))\,dt,$$

where $\mathcal{E}[\lambda](t)$ is a $r \times r \times r-$ tensor with Lipschitzian time-variable entries, linearly depending on λ. Next we introduce the functional:

$$L[\lambda](y) = \int_0^T [(Q[\lambda](t)y, y) + (\mathcal{E}[\lambda](t)y, y, u)]\,dt, \qquad (11)$$

where $Q[\lambda](t)$ is the matrix from the second variation of Lagrange function, corresponding to λ, being presented in the form (8).

The additional, new Legendre type condition, relevant to Pontryagin minimum, is as follows. Choose any time instant t_* , and freeze all coefficients in (11) at this point, so that we get the functional

$$L[\lambda, t_*](y) = \int_0^T [(Q[\lambda](t_*)y, y) + (\mathcal{E}[\lambda](t_*)y, y, u)]\,dt.$$

Consider all absolutely continuous functions $y(t)$, having zeros at both endpoints of the interval, and $\dot y = u \in U(t_*)$ (the admissible control set, frozen at t_*). Then, for any such function the following inequality must hold:

$$L[\lambda, t_*](y) \;\ge\; a\int_0^T (y, y)\,dt, \qquad (12)$$

where, again, $a = 0$ for the necessary conditions, and $a > 0$ for the sufficient ones.

If we denote the above set $M(\Lambda)$, corresponding to the weak minimum, by $G(\Lambda)$, then the set $M(\Lambda)$, corresponding to the $\Pi-$ minimum, consists of all $\lambda \in G(\Lambda)$, such that inequality (12) holds for each $t_* \in [0, T]$, and for any $y(t)$ with the above properties. To be more precise, for the $\Pi-$ minimum we have, actually, not a single set $M(\Lambda)$, but a family $M_a(\Lambda) \subset G(\Lambda)$, $a \in R$. The set $M_0(\Lambda)$ comes into the necessary condition (2) for the $\Pi-$ minimum, and the set $M_a(\Lambda)$, for some $a > 0$, comes into the sufficient condition (3).

Condition (12) has intrinsic interest, and can be investigated independently of the initial problem, and separately for any given λ and t_* . Thus, to study the nature of this condition, one can simply consider the functional:

$$L(y) = \int_0^T [(Qy, y) + (\mathcal{E}y, y, u)]dt,$$

where Q is a constant matrix, and $\mathcal{E}$ is a constant tensor, and solve the following auxiliary problem: to find the maximal real a such that for any absolutely continuous function $y(t)$, having $y(0) = y(T) = 0$, and $\dot{y} = u \in U$, the following inequality holds:

$$L(y) = \int_0^T [(Qy, y) + (\mathcal{E}y, y, u)]dt \;\geq\; a \int_0^T (y, y)dt. \tag{13}$$

It can be easily noticed, that the fulfilment of (13) does not depend on the length of interval, so one may consider it on any interval, e.g. on [0,1].

At present we can solve this problem completely for three cases of the control set U : a) the whole space, b) an arbitrary (hyper-)stripe, c) an arbitrary ellipse on the plane. (Both the stripe and the ellipse must contain the origin in their interiors.)

4 Particular cases

4.1 U is the whole space

Begin with the most simple case (a), in which u is unconstrained. Here condition (13) decomposes on two conditions, concerning Q and $\mathcal{E}$ separately. Define the differential 1-form

$$\omega \;=\; (\mathcal{E}y, y, dy) \;=\; \sum_{ijk} \mathcal{E}_{ijk}\, y^i y^j dy^k \;.$$

Theorem 2 [11–15]. *Condition (13) holds for a given* a *iff* $Q \geq a$ *(Goh condition for a weak minimum), and* ω *is closed, i.e.*

$$d\omega \;=\; \sum_{ijk} \mathcal{E}_{ijk}\,(y^i dy^j + y^j dy^i) \wedge dy^k = 0. \tag{14}$$

Thus, in this case we obtain an additional optimality condition of equality type. Equation (14), as well as Goh conditions, can be expressed in terms of Lie brackets for initial system (1). Let for simplicity f and F are independent of t, so that the system (1) is of the form:

$$\dot{x} = f_0(x) + \sum_{i=1}^{r} u_i f_i(x),$$

where $f_i(x)$ are the columns of the matrix $F(x)$. Here the Goh equality condition $V(t) = 0$ takes the form:

$$\psi(t)\,[f_i, f_k] = 0 \qquad \text{for all} \quad i,k = 1,\dots,r \tag{15}$$

along the reference trajectory $x^0(t)$; the matrix Q has elements

$$Q_{ik} = \psi(t)\,[\,[f_i, f_0], f_k\,], \qquad i,k = 1,\dots,r, \tag{16}$$

and condition (14) means that along $x^0(t)$

$$\psi(t)\,[\,[f_i, f_j], f_k\,] = 0 \qquad \text{for all} \quad i,j,k = 1,\dots,r. \tag{17}$$

Remark. If $\mathcal{E}(t)$ is a time-dependent tensor with Lipschitz entries, and for each t_* the tensor $\mathcal{E} = \mathcal{E}(t_*)$ satisfies condition (14), then there exists a Lipschitz tensor $T(t)$ of the same dimensions, such that $\forall\, t_* \quad \omega = d\varphi$, where $\varphi = (T(t_*)y, y, y)$ is a scalar cubic function. Hence, for any $y(t)$

$$\int_0^T (\mathcal{E}(t)y, y, u))\,dt \;=\; (T(t)y, y, y)\Big|_0^T \;-\; \int_0^T (\dot{T}(t)y, y, y)\,dt. \tag{18}$$

Now, if one take an arbitrary sequence $y_n(t)$, such that $||y_n||_C \to 0$, then, obviously, the right hand side in (18) is $o\,(\gamma(y_n))$, whence the left hand side is $o\,(\gamma(y_n))$ too. Thus, in this case for any Pontryagin sequence the whole functional $\rho(w_n)$ is $o\,(\gamma(w_n))$, and so we do not need to take it into account.

4.2 The set U is a stripe

We consider the case of hyper-stripe, i.e. when the set U is given by inequalities $-b \le (l,u) \le c$, the l being an arbitrary vector in R^r, and $b, c > 0$. In an appropriate basis such a stripe takes the form $-b \le u_1 \le c$, where u_1 is the first component of u, and the rest components are free.

Let us establish the following property of $L(y)$.

Lemma 1. *If condition (13) holds with some $a \in R$, then $L(y)$ can be reduced to the form:*

$$L = \int (Q(y) + E(y)u_1)\, dt, \tag{19}$$

where Q and E are quadratic forms of y .

Proof. We have to show, that the terms $\int P_i(y)u_i\, dt$, $i = 2, \ldots, r$, where P_i are quadratic forms of y, can be reduced to the term $\int E(y)u_1\, dt$.

If P_i depends only on y_1, i.e. $P_i(y)u_i = p_i\, y_1^2\, u_i$, then, using the integration by parts, we come to the term $2\int p_i\, y_i\, y_1\, u_1\, dt$, what is just needed.

Next, the restriction of L to the subspace $y_1 = 0$ still satisfy (13), and since $u_2, \ldots, u_r$ are free, by Theorem 2 the restriction of the 1-form ω to the hyperplane $y_1 = 0$ must be closed, whence all the terms in L, containing neither y_1 nor u_1, give zero integral, and hence can be neglected.

Thus, it remains to consider only the term $\int y_1(S'y', u')\, dt$, where $y' = (y_2, \ldots, y_r)$, $u' = (u_2, \ldots, u_r)$, and S is a constant $(r-1) \times (r-1)-$ matrix. Now, fix any admissible nonzero function $y_1(t)$, and denote $S'(t) = y_1(t)S$. Then the above term is $\int (S'(t)y', u'\, dt$, where u' is free, and from (13) and Goh equality condition (9a) it readily follows, that the matrix $S'(t)$ is symmetric, which obviously implies that S is symmetric as well. But in this case $(Sy', u') = \frac{1}{2}\frac{d}{dt}(Sy'\, y')$, and so, integrating by parts the above term, we get $-\int \frac{1}{2}(S'y', y')\, u_1\, dt$, which is of the required form. Lemma is proved.

Due to this lemma, one can consider the functional L in the form (19), where $u = (u_1, u_2)$, $\dim\ u_1 = 1$, $\dim\ u_2 = r-1$, $y = (y_1, y_2)$, $\dot{y} = u$, $y(0) = 0$, and Q, E are quadratic forms of y .

Theorem 3 (A.A. Milyutin). *Condition (13) with $a = 0$ holds for the functional (19) iff $Q \ge 0$, and for any vectors $y_1,\, y_2',\, y_2''$ of dimensions $1, r-1, r-1$ respectively, the following inequality holds:*

$$\frac{1}{c}\, Q(y_1, y_2') \,+\, E(y_1, y_2') \,+\, \frac{1}{b}\, Q(y_1, y_2'') \,-\, E(y_1, y_2'') \,\ge\, 0.$$

Observe that here in the left hand side we have simply a finite-dimensional quadratic form, so the question of verification for condition (13) is reduced to a standard question of linear algebra.

4.3 U is an ellipse on the plane,

containing the origin in its interior. Let its dual ellipse be given by the inequality: $(S(u-q),(u-q)) \le 1$, where S is a symmetric positive definite matrix, and $(Sq,q) < 1$. We assume here that the matrix Q is positive definite too (which is necessary for the fulfilment of (13) with $a > 0$.) Then L can be reduced to the form:

$$L = \int [\,(y,y) - (E^1 y, y)\, u_1 - (E^2 y, y)\, u_2\,]\, dt,$$

where E^1, E^2 are arbitrary constant matrices. Consider the differential 1-form

$$\omega = \, (E^1 y, y)\, dy_1 + (E^2 y, y)\, dy_2$$

and compute $d\omega$, which can be presented as

$$d\,\omega = \, 2\,(l_1 y_1 + l_2 y_2)\, dy_1 \wedge dy_2.$$

Denote $l = (l_1, l_2)$, $G = S^{-1/2}$, and let P be the matrix of rotation through the angle 90^o.

Note that if we take another 1-form ω' (i.e. another matrices E^1 and E^2), such that $d\,(\omega' - \omega) = 0$, then obviously L would not change. Therefore, in order to obtain a rough estimate of the maximal a, we can proceed as follows. Suppose that $\forall\, u \in U$ and $\forall\, y \in R^r$ the following estimate holds:

$$|\,(E^1 y, y)\, u_1 + (E^2 y, y)\, u_2\,| \;\le\; \mu(y,y). \tag{20}$$

(If U is bounded, such a μ does exist.) Then, obviously, $L(y) \ge (1-\mu) \int (y,y) dt$, i.e. inequality (13) holds with $a = 1 - \mu$. However, this is only one possible a, while we try to find the maximal a. The interesting fact is that, for the case when U is an ellipse, if one chooses appropriate matrices E^1, E^2 (with the same $d\omega$), one can obtain the maximal a . Thus, we have to consider all possible pairs E^1, E^2, generating the given $d\omega$, for each pair to calculate μ from (20), put $a = 1 - \mu$, and take $\max\{a\}$. This will be the exact value.

This procedure results in the following

Theorem 4. *The maximal* a, *for which condition (13) holds, is equal to* $1-\mu$, *where*

$$\mu \;=\; \frac{\mid l\mid^2}{(q,Pl)-\sqrt{(TrG)^2\mid l\mid^2-(q,l)^2}}\,.$$

In particular, if U *is a centrally symmetric ellipse with the axis* $2b_1,\, 2b_2$, *then*

$$\mu \;=\; \frac{b_1 b_2}{b_1+b_2}\mid l\mid .$$

The proof is rather not obvious; it is based not on optimality conditions, but on the following

Theorem 5. *Let* $f:\, R^2\to R^2$ *be a positive homogeneous function of zero degree,* $f\neq 0$ *and Lipschitz continuous outside of the origin. Suppose that when a number* θ *passes an interval of length* 2π, *the argument of* $f(\exp(i\theta))$ *increases in* 4π. *Then equation* $\dot y = f(y)$ *has a cyclic solution with zero endpoints on some interval.*

The idea of the above procedure can be applied for the general case, when U is an arbitrary convex compactum in R^r with $0\in \operatorname{int} U$, but the resulting formulas are more complicated. They have a deep relation with the problem of approximation of an arbitrary vector field on an open set in R^r by gradient vector fields; see [18].

5 Other properties

Condition (12) is in a sense invariant w.r.t. a change of control variables. Consider two systems:

$$\dot x = f(x,t)+F(x,t)u, \qquad u\in U(x,t), \tag{21}$$

$$\dot x = f(x,t)+G(x,t)v, \qquad v\in V(x,t), \tag{22}$$

under the assumptions that in a neigborhood of a point (x_*,t_*) rank $F(x,t)$ = rank $G(x,t)$ = const, and moreover $F(x,t)U(x,t)=G(x,t)V(x,t)$.

This means that in the state space both systems (21) and (22) have the same capability, and can be transformed one to another by a change of the control variables.

Theorem 6. *Suppose that at a point* (ψ_*,x_*,t_*) *condition (12) holds with* $a>0$ $(a=0)$ *for system (21). Then it holds with some* $a'>0\;(a=0)$ *for system (22) as well.*

The proof is based on a nontrivial interpretation of condition (12) as a property of Lagrange function on a special class of local variations, and on a recently developed invariance theory of extremals [17].

Condition (14) have relations to the known procedure of determination singular extremals (consisting of successive differentiations of the switching function), and to the existence of a so-called Krotov function, which is often used as the basic object in formulation of sufficient conditions for a strong minimum [16].

6 Example

Let $x \in R^2,\ u \in R^2, \quad t \in [0,1], \qquad \dot{x}_1 = u_1, \quad \dot{x}_2 = u_2 - bx_2u_1, \qquad x(0) = 0,$

$$J = \int_0^1 (2x_1u_1 + 2x_2u_2 + x_1^2 + x_2^2)\, dt \to \min.$$

Here b is a parameter, the examined trajectory is $w^0 = (x^0, u^0) \equiv (0,0)$, and let us first consider the case when u is free, i.e. $U = R^2$.

The critical cone $\mathcal{K}$ is given by equalities $\dot{x} = u,\ x(0) = 0$, the Lagrange multipliers are unique, the second variation of Lagrange function $\Omega = J = \gamma$ on $\mathcal{K}$, where

$$\gamma(y) = |\ y_1(1)\ |^2 + |\ y_2(1)\ |^2 + \int (|y_1(t)|^2 + |y_2(t)|^2)\, dt.$$

Obviously, Ω satisfies Goh conditions (9) and the sufficient condition (3) for the weak minimum, whence for any b the w^0 is a weak minimum point in the problem.

However, if we consider the cubic functional

$$\rho(w) = \ 2b \int y_2^2\, u_1\, dt \ + \ o(\gamma),$$

and compute

$$\omega = 2b\, y_2^2\, dy_1 \quad \text{and} \qquad d\omega = -4b\, y_2\, dy_1 \wedge dy_2,$$

then it is clear, that condition (14) is fulfilled only for $b = 0$, which implies that $\Pi-$ minimum at w^0 holds if and only if $b = 0$.

Let us take $b \neq 0$ and restrict the control by the constraint $|u| \le r$ (a ball). For which r there will be Π -minimum? The presence of the weak minimum at w^0 ensures us, that for all small enough $r > 0$ there is also $\Pi-$ minimum at w^0. Condition (13) and Theorem 4 allow us to determine the maximal value of these r. Here the result is: $\max a = 1 - r \mid b \mid$, so

if $r \mid b \mid < 1$, then Π -minimum holds,
and if $r \mid b \mid > 1$, then Π -minimum fails to hold.

Consider the same initial problem with another constraint: $|u_1| \le r$ (a stripe). Here $\max a = 1 - 2r\,|b|$, so

$$\begin{array}{lll} \text{if} & 2r\,|b| < 1, & \text{then } \Pi\text{-minimum holds,} \\ \text{and if} & 2r\,|b| > 1, & \text{then } \Pi\text{-minimum fails to hold.} \end{array}$$

Note that the critical value of r for the stripe is two times less than that for the circle.

Finally, consider an ellipse:

$$\left(\frac{u_1}{r_1}\right)^2 + \left(\frac{u_2}{r_2}\right)^2 \le 1.$$

Here the critical size for $\Pi-$ minimum is defined by the relation:

$$2\,\frac{r_1 r_2}{r_1 + r_2}\,|b| = 1.$$

Note that if $r_1 = r_2$, it is reduced to the case of the circle, and if $r_2 \to \infty$, we get precisely the critical value for the stripe.

7 Acknowledgements

This work was supported by the Russian Foundation for Basic Research under grants 96-15-96072 and 97-01-00135.

References

[1] H.J.Kelley, R.E.Kopp, H.G.Moyer, "Singular extremals", - in *Topics in Optimization* (ed. G.Leitman), Acad. Press, New York–London, p. 63–101, (1967).

[2] B.S.Goh, "Necessary conditions for singular extremals involving multiple control variables", *SIAM J. on Control,* **4**, No. 4, p. 716–731, (1966).

[3] R.Gabasov, F.M.Kirillova, "Singular optimal controls", Nauka, Moscow, (1973).

[4] D.J.Bell, D.H.Jacobson, "Singular Optimal Control Problems", Academic Press, NY, (1975).

[5] H.W.Knobloch, "Higher order necessary conditions in optimal control theory", *Lecture Notes in Control and Inf. Sci.,* **34**, (1981).

[6] A.J.Krener, "The high order maximal principle and its application to singular extremals", *SIAM J. on Control,* **15**, No. 2, p. 256–293, (1977).

[7] A.A.Agrachiov, R.V.Gamkrelidze, "Second order optimality principle for a time-optimal problem", *Math. USSR, Sbornik,* **100**, No. 4, (1976).

[8] M.I.Zelikin, "An optimality condition for singular trajectories in the problem of minimization of a curvilinear integral", *Soviet Math. Doklady*, **267**, No. 3, (1982) (in Russian).

[9] F.Lamnabhi-Lagarrigue, G.Stefani, "Singular optimal control problems: on the necessary conditions of optimality", *SIAM J. on Control & Optim.*, **28**, No. 4, p. 823–840, (1990).

[10] A.A.Milyutin, "Quadratic extremum conditions in smooth problems with a finite-dimensional image", - in *Metody teorii ekstremal'nyh zadach v ekonomike*, Nauka, Moscow, p. 138–177, (1981) (in Russian).

[11] A.V.Dmitruk, "Quadratic conditions for a Pontryagin minimum in an optimal control problem, linear in the control, with a constraint on the control", *Soviet Math. Doklady*, **28**, No. 2, p. 364–368, (1983).

[12] A.V.Dmitruk, "Quadratic conditions for a Pontryagin minimum in an optimal control problem, linear in the control", *Mathematics of the USSR, Izvestija*, **28**, No. 2, p. 275–303, (1987), and **31**, No. 1, p. 121–141, (1988).

[13] A.V.Dmitruk, "Second order necessary and sufficient conditions of a Pontryagin minimum for singular regimes", *Lecture Notes in Control and Inf. Sciences*, **180**, p. 334–343, (1992).

[14] A.V.Dmitruk, "Second order optimality conditions for singular extremals", - in *Computational Optimal Control* (R.Bulirsch and D.Kraft eds.), Internat. Ser. on Numer. Mathematics, Birkhaueser, Basel, **115**, p. 71–81, (1994).

[15] A.V.Dmitruk, "Second order necessary and sufficient conditions of a Pontryagin minimum for singular boundary extremals", *Proc. of Internat. Congress on Industrial and Applied Math. in Hamburg,* ZAMM, Issue 3, p. 411–412, (1995).

[16] A.V.Dmitruk, "On the problem of necessity of the Krotov type sufficient conditions", *Automatics and Remote Control*, No. 10, (1997).

[17] A.A.Milyutin, "Invariance theory of extremals", - in *Necessary Condition in the Optimal Control* (A.A.Milyutin ed.), Nauka, Moscow, p. 81–131, (1990) (in Russian).

[18] A.A.Milyutin, "On a duality formula for multidimensional vector fields", *Russian Journal of Mathematical Physics*, **3**, No. 1, p. 81–112, (1995).

Published in "Calculus of Variations and Optimal Control", Chapman & Hall/CRC Res. Notes in Math. Series, Vol. 411, CRC Press, Boca Raton, FL, 1999, p. 49–61.